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ELEMENTARY ALGEBRA

BY

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REVISED AND ENLARGED

FOR THE USE OF AMERICAN SCHOOLS

BY

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PREFACE.

WITHIN a comparatively short time, the algebra requirement for admission to many of our Colleges and Schools of Science has been much increased in both thoroughness of preparation and amount of subject-matter. This increase has made necessary the rearrangement and extension of elementary algebra, and it is for this reason that the present revision of Hall and Knight's *Elementary Algebra* has been undertaken.

The marked success of the work, and the hearty endorsement by many of our ablest educators of the treatment of the subject as therein presented, warrant the belief that the present edition, with its additional subject-matter, will be found a desirable arrangement and satisfactory treatment of every part of the subject required for admission to any of our Colleges or Schools of Technology.

Many changes in the original chapters have been made, among which we would call attention to the following: A proof, by mathematical induction, of the binomial theorem for positive integral index has been added to Chapter XXXIX.; a method of finding a factor that will rationalize any binomial surd follows the treatment of binomial quadratic surds; Chapter XLII. has been re-written in part, and appears as a chapter on equations in quadratic form; and the chapter on logarithms has been enlarged by the addition of a four-place table of logarithms with explanation of its use.

Chapters XXI., XXV., XXX., XXXIII., XXXVIII., XLII., XLIII., XLIV., XLV., XLVI., XLVII., XLIX., and L. treat of portions of the subject that have not appeared in former editions. A chapter on General Theory of Equations is not usually found in an Elementary Algebra, but properly finds here a place in accordance with the purpose of the present revision; and its introduction makes the work available for use in college classes. Carefully selected exercises are given with each chapter, and at the end of the work a large miscellaneous collection will be found.

The *Higher Algebra* of Messrs. Hall and Knight has been drawn upon, and the works of Todhunter, Chrystal, and DeMorgan consulted in preparing the new chapters.

I gratefully acknowledge my indebtedness to Prof. J. Burkitt Webb of the Stevens Institute of Technology both for contributions of subject-matter, and valuable suggestions as to methods of treatment. My thanks are also due to Prof. W. H. Bristol, of the same institution, for suggestions as to the arrangement of the chapter on General Theory of Equations.

FRANK L. SEVENOAK.

JUNE, 1895.

PREFACE TO SECOND EDITION.

The printing of the present edition from entirely new plates has enabled us to correct a few typographical errors found in the first edition, and give, at the suggestion of friends, a somewhat fuller explanation of the more difficult parts of the subject. We hope that the addition of new material to Chapters III., IV., V., X., XX., XXI., XLII., XLVIII., and of several sets of Miscellaneous Examples, will render the book still more acceptable to those whose commendation of the former edition has given us much pleasure.

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ALGEBRA.

CHAPTER I.

Definitions. Substitutions.

1. Algebra treats of quantities as in Arithmetic, but with greater generality; for while the quantities used in arithmetical processes are denoted by *figures*, which have a single definite value, algebraic quantities are denoted by *symbols*, which may have any value we choose to assign to them.

The symbols of quantity employed are usually the letters of our own alphabet; and, though there is no restriction as to the numerical values a symbol may represent, it is understood that in the same piece of work it keeps the same value throughout. Thus, when we say "let a equal 1," we do not mean that a must have the value 1 always, but only in the particular example we are considering. Moreover, we may operate with symbols without assigning to them any particular numerical value; indeed it is with such operations that Algebra is chiefly concerned.

We begin with the definitions of Algebra, premising that the symbols $+$, $-$, $\times$, $\div$, $(\,)$, $=$ will have the same meanings as in Arithmetic. Also, for the present, it will be assumed that all the algebraic symbols employed represent integral numbers.

2. An **algebraic expression** is a collection of symbols; it may consist of one or more **terms**, which are the parts separated from each other by the signs + and −. Thus, $7a + 5b - 3c - x + 2y$ is an expression consisting of five terms.

NOTE. When no sign precedes a term the sign + is understood.

3. **Expressions** are either **simple** or **compound.** A *simple expression* consists of *one* term, as $5a$. A *compound expression* consists of *two or more* terms. Compound expressions may be further distinguished. Thus an expression of *two* terms, as $3a - 2b$, is often called a **binomial,** and one of *three* terms, as $2a + 3b + c$, a **trinomial.** Simple expressions are frequently spoken of as **monomials,** and compound expressions as **multinomials** or **polynomials.**

4. When two or more quantities are multiplied together the result is called the **product.** One important difference between the notation of Arithmetic and Algebra should be here remarked. In Arithmetic the product of 2 and 3 is written 2×3, whereas in Algebra the product of a and b may be written in any of the forms $a \times b$, $a \cdot b$, or ab. The form ab is the most usual. Thus, if $a = 2$, $b = 3$, the product $ab = a \times b = 2 \times 3 = 6$; but in Arithmetic 23 means "twenty-three," or $2 \times 10 + 3$.

5. Each of the quantities multiplied together to form a product is called a **factor** of the product. Thus 5, a, b are the factors of the product $5ab$.

NOTE. The beginner should carefully notice the difference between **term** and **factor.**

6. When one of the factors of an expression is a numerical quantity, it is called the **coefficient** of the remaining factors. Thus, in the expression $5ab$, 5 is the coefficient. But the word coefficient is also used in a wider sense, and it is sometimes convenient to consider any factor, or factors, of a product as the coefficient of the remaining factors. Thus, in the product $6abc$, $6a$ may be appropriately called the coefficient of bc. A coefficient which is not merely numerical is sometimes called a **literal coefficient.**

NOTE. When the coefficient is unity it is usually omitted, and we write simply a, instead of $1a$.

7. A **power** of a quantity is the product obtained by repeating that quantity any number of times as a factor, and is expressed by writing the number of factors to the right of the quantity and above it. Thus,

$a \times a$ is called the *second power* of a, and is written a^2;

$a \times a \times a$ is called the *third power* of a, and is written a^3;

and so on.

The **index** or **exponent** is the number which expresses the power of any quantity. Thus 2, 5, 7 are respectively the indices of a^2, a^5, a^7.

NOTE. a^2 is usually read "a squared"; a^3 is read "a cubed"; a^4 is read "a to the fourth"; and so on.

When the index is unity it is omitted, and we write simply a, instead of a^1. Thus a, $1a$, a^1, and $1a^1$ all have the same meaning.

8. The beginner must be careful to distinguish between *coefficient* and *index*.

Ex. **1.** What is the difference in meaning between $3a$ and a^3?

By $3a$ we mean the product of the quantities 3 and a.

By a^3 we mean the product of the quantities a, a, a.

Thus, if $a = 4$,

$$3a = 3 \times a = 3 \times 4 = 12;$$
$$a^3 = a \times a \times a = 4 \times 4 \times 4 = 64.$$

Ex. **2.** If $b = 5$, distinguish between $4b^2$ and $2b^4$.

Here $4b^2 = 4 \times b \times b = 4 \times 5 \times 5 = 100$;

whereas $2b^4 = 2 \times b \times b \times b \times b = 2 \times 5 \times 5 \times 5 \times 5 = 1250$.

Ex. **3.** If $x = 1$, find the value of $5x^4$.

Here $5x^4 = 5 \times x \times x \times x \times x = 5 \times 1 \times 1 \times 1 \times 1 = 5$.

NOTE. **The beginner should observe that every power of 1 is 1.**

Ex. **4.** If $a = 4$, $x = 1$, find the value of $5x^a$.

$$5x^a = 5 \times x^a = 5 \times 1^4 = 5 \times 1 = 5.$$

9. The **Sign of Continuation**, $\cdots$, is read "**and so on.**"

10. The **Sign of Deduction**, $\therefore$, is read "**therefore**" or "**hence.**"

11. In arithmetical multiplication the order in which the factors of a product are written is immaterial. Thus

$$3 \times 4 = 4 \times 3.$$

In like manner in Algebra ab and ba each denote the product of the two quantities represented by the letters a and b, and have therefore the same value. Although it is immaterial in what order the factors of a product are written, it is usual to arrange them alphabetically. Fractional coefficients which are greater than unity are usually kept in the form of improper fractions.

Ex. If $a = 6$, $x = 7$, $z = 5$, find the value of $\frac{13}{10}axz$.

Here $\frac{13}{10}axz = \frac{13}{10} \times 6 \times 7 \times 5 = 273$.

EXAMPLES I. a.

If $a = 7$, $b = 2$, $c = 1$, $x = 5$, $y = 3$, find the value of

1. $14x$.	**3.** $3ax$.	**5.** $5by$.	**7.** $3b^2$.	**9.** $6c^4$.
2. x^3.	**4.** a^3.	**6.** b^5.	**8.** $2ax$.	**10.** $4y^3$.

If $a = 8$, $b = 5$, $c = 4$, $x = 1$, $y = 3$, find the value of

11. $3c^2$.	**14.** $9xy$.	**17.** x^8.	**20.** b^y.	**23.** y^b.
12. $7y^3$.	**15.** $8b^3$.	**18.** $7y^4$.	**21.** y^c.	**24.** a^y.
13. $5ab$.	**16.** $3x^5$.	**19.** c^x.	**22.** x^b.	**25.** b^x.

If $a = 5$, $b = 1$, $c = 6$, $x = 4$, find the value of

26. $\frac{1}{4}x^4$.	**28.** $\frac{3}{8}x^3$.	**30.** 3^x.	**32.** 8^b.	**34.** $\frac{7}{15}acx$.
27. $\frac{5}{12}c^3$.	**29.** $\frac{1}{10}ax$.	**31.** 2^c.	**33.** 7^x.	**35.** $\frac{1}{8}bcx$.

12. When several different quantities are multiplied together a notation similar to that of Art. 7 is adopted. Thus $aabbbbcddd$ is written $a^2b^4cd^3$. And conversely $7a^3cd^2$ has the same meaning as $7 \times a \times a \times a \times c \times d \times d$.

Ex. 1. If $x = 5$, $y = 3$, find the value of $4x^2y^3$.

$$4x^2y^3 = 4 \times 5^2 \times 3^3 = 4 \times 25 \times 27 = 2700.$$

Ex. 2. If $a = 4$, $b = 9$, $x = 6$, find the value of $\frac{8bx^2}{27a^3}$.

$$\frac{8bx^2}{27a^3} = \frac{8 \times 9 \times 6^2}{27 \times 4^3} = \frac{8 \times 9 \times 36}{27 \times 64} = \frac{3}{2} = 1\tfrac{1}{2}.$$

13. If one factor of a product is equal to 0, the product must be equal to 0, *whatever values the other factors may have.* A factor 0 is usually called a *zero factor.*

For instance, if $x = 0$, then ab^3xy^2 contains a zero factor. Therefore $ab^3xy^2 = 0$ when $x = 0$, whatever be the values of a, b, y.

Again, if $c = 0$, then $c^3 = 0$; therefore $ab^2c^3 = 0$, whatever values a and b may have.

NOTE. **Every power of 0 is 0.**

EXAMPLES I. b.

If $a = 7$, $b = 2$, $c = 0$, $x = 5$, $y = 3$, find the value of

1. $4ax^2$.
2. a^3b.
3. $8b^2y$.
4. $3xy^2$.
5. $\frac{3}{4}b^2x$.
6. $\frac{5}{6}b^3y^2$.
7. $\frac{2}{5}xy^4$.
8. a^3c.
9. a^2cy.
10. $8x^3y$.

If $a = 2$, $b = 3$, $c = 1$, $p = 0$, $q = 4$, $r = 6$, find the value of

11. $\frac{3a^2r}{8b}$.
12. $\frac{8ab^2}{9q^2}$.
13. $\frac{6a^3c}{b^2}$.
14. $\frac{4cr^2}{9a^3}$.
15. $3a^2b^c$.
16. $\frac{5}{6}ba^r$.
17. $\frac{8b}{9a^r}$.
18. $5a^bc^r$.
19. $\frac{2a^2p}{7r}$.
20. 3^a2^b.
21. 2^ra^5.
22. c^bb^q.
23. $\frac{5a^rb^q}{64r^a}$.
24. $\frac{27a^q}{32}$.
25. $\frac{64}{q^r}$.

14. DEFINITION. The **square root** of any proposed expression is that quantity whose square, or second power, is equal to the given expression. Thus the square root of 81 is 9, because $9^2 = 81$.

The square root of a is denoted by $\sqrt[2]{a}$, or more simply $\sqrt{a}$.

Similarly the **cube, fourth, fifth,** etc., **root** of any expression is that quantity whose third, fourth, fifth, etc., power is equal to the given expression.

The roots are denoted by the symbols $\sqrt[3]{}$, $\sqrt[4]{}$, $\sqrt[5]{}$, etc.

EXAMPLES: $\sqrt[3]{27} = 3$; because $3^3 = 27$.

$\sqrt[5]{32} = 2$; because $2^5 = 32$.

The symbol $\sqrt{}$ is sometimes called the **radical sign.**

Ex. 1. Find the value of $5\sqrt{(6a^3b^4c)}$, when $a=3,\ b=1,\ c=8$.

$$\begin{aligned} 5\sqrt{(6a^3b^4c)} &= 5\times\sqrt{(6\times 3^3\times 1^4\times 8)} \\ &= 5\times\sqrt{(6\times 27\times 8)} = 5\times\sqrt{1296} \\ &= 5\times 36 = 180. \end{aligned}$$

Ex. 2. Find the value of $\sqrt[3]{\left(\dfrac{ab^4}{8x^3}\right)}$, when $a=9,\ b=3,\ x=5$.

$$\sqrt[3]{\left(\frac{ab^4}{8x^3}\right)} = \sqrt[3]{\left(\frac{9\times 3^4}{8\times 5^3}\right)} = \sqrt[3]{\left(\frac{9\times 81}{8\times 125}\right)} = \sqrt[3]{\left(\frac{9\times 9\times 9}{1000}\right)} = \frac{9}{10}.$$

EXAMPLES I. c.

If $a=8,\ c=0,\ k=9,\ x=4,\ y=1$, find the value of

1. $\sqrt{(2a)}$.
2. $\sqrt{(kx)}$.
3. $\sqrt{(2ax)}$.
4. $\sqrt{(2ak^2)}$.
5. $\sqrt[3]{(3k)}$.
6. $\sqrt[3]{(ax^3)}$.
7. $\sqrt[3]{(8x^3y^3)}$.
8. $\sqrt[3]{(cy^5)}$.
9. $2x\sqrt{(2ay)}$.
10. $5y\sqrt{(4kx)}$.
11. $3c\sqrt{(kx)}$.
12. $2xy\sqrt{(4y^5)}$.
13. $\sqrt{\left(\dfrac{8x^3}{ak}\right)}$.
14. $\sqrt{\left(\dfrac{25a}{2k}\right)}$.
15. $\sqrt{\left(\dfrac{16x}{49y^3}\right)}$.
16. $\sqrt{\left(\dfrac{ca^2}{16k}\right)}$.
17. $\sqrt[3]{\left(\dfrac{3a}{k^2}\right)}$.
18. $\sqrt[3]{\left(\dfrac{ax^3}{27y^3}\right)}$.

If $a=4,\ b=1,\ c=2,\ d=9,\ x=5,\ y=8$, find the value of

19. $\sqrt{(8ac)}$.
20. $6\sqrt{(4b^3)}$.
21. $7\sqrt{(5dx)}$.
22. $\sqrt{(c^5y)}$.
23. $\sqrt{\left(\dfrac{1}{5xy^2}\right)}$.
24. $\sqrt{\left(\dfrac{1}{8acd}\right)}$.
25. $\dfrac{1}{\sqrt{(9a^3c^2)}}$.
26. $\dfrac{1}{\sqrt{(5c^4x)}}$.
27. $\sqrt[3]{\left(\dfrac{1}{4cx^3}\right)}$.
28. $\dfrac{1}{\sqrt[3]{(8ab^2c)}}$.
29. $\sqrt[3]{\left(\dfrac{3x^3}{d^2y}\right)}$.
30. $\sqrt[3]{\left(\dfrac{5b^8}{x^4y}\right)}$.
31. $\sqrt{d^c}$.
32. $\sqrt{y^a}$.
33. $\sqrt{b^d}$.
34. $\sqrt{d^b}$.

15. We now proceed to find the numerical value of expressions which contain more than one term. In these, each term can be dealt with singly by the rules already given, and by combining the terms the numerical value of the whole expression is obtained.

16. We have already, in Art. 8, called attention to the importance of carefully distinguishing between *coefficient* and *index;* confusion between these is such a fruitful source of error with beginners that it may not be unnecessary once more to dwell on the distinction.

Ex. 1. When $c = 5$, find the value of $c^4 - 4c + 2c^3 - 3c^2$.

Here
$$c^4 = 5^4 = 5 \times 5 \times 5 \times 5 = 625;$$
$$4c = 4 \times 5 = 20;$$
$$2c^3 = 2 \times 5^3 = 2 \times 5 \times 5 \times 5 = 250;$$
$$3c^2 = 3 \times 5^2 = 3 \times 5 \times 5 = 75.$$

Hence the value of the expression $= 625 - 20 + 250 - 75 = 780$.

Ex. 2. When $p = 9$, $r = 6$, $k = 4$, find the value of
$$\frac{1}{3}\sqrt[3]{\left(\frac{pr}{k^2}\right)} + \sqrt{(3k + k^3 + 5)} - \frac{2r^2}{9k}.$$

$$\frac{1}{3}\sqrt[3]{\left(\frac{pr}{k^2}\right)} + \sqrt{(3k + k^3 + 5)} - \frac{2r^2}{9k} = \frac{1}{3}\sqrt[3]{\left(\frac{54}{16}\right)} + \sqrt{(12+64+5)} - \frac{2\times 36}{9\times 4}$$
$$= \tfrac{1}{3}\sqrt[3]{\tfrac{27}{8}} + \sqrt{81} - 2$$
$$= \tfrac{1}{3} \times \tfrac{3}{2} + 9 - 2 = 7\tfrac{1}{2}.$$

17. By Art. 13 any term which contains a *zero factor* is itself zero, and may be called a *zero term.*

Ex. 1. If $a = 2$, $b = 0$, $x = 3$, $y = 1$, find the value of
$$4a^3 - ab^3 + 2xy^2 + 3abx.$$

The expression
$$= (4 \times 2^3) - 0 + (2 \times 3 \times 1) + 0$$
$$= 32 - 0 + 6 + 0 = 38.$$

NOTE. The two *zero terms* do not affect the result.

Ex. 2. Find the value of $\frac{3}{5}x^2 - a^2y + 7abx - \frac{5}{2}y^3$, when
$$a = 5,\ b = 0,\ x = 7,\ y = 1.$$
$$\tfrac{3}{5}x^2 - a^2y + 7abx - \tfrac{5}{2}y^3 = \tfrac{3}{5} \times 7^2 - 5^2 \times 1 + 0 - \tfrac{5}{2} \times 1^3$$
$$= 29\tfrac{2}{5} - 25 - 2\tfrac{1}{2} = 1\tfrac{9}{10}.$$

NOTE. The *zero term* does not affect the result.

18. In working examples the student should pay attention to the following hints:

1. Too much importance cannot be attached to neatness of style and arrangement. The beginner should remember that neatness is in itself conducive to accuracy.

2. The sign = should never be used except to connect quantities which are equal. Beginners should be particularly careful not to employ the sign of equality in any vague and inexact sense.

3. Unless the expressions are very short the signs of equality in the steps of the work should be placed one under the other.

4. It should be clearly brought out how each step follows from the one before it; for this purpose it will sometimes be advisable to add short verbal explanations; the importance of this will be seen later.

EXAMPLES I. d.

If $a = 2$, $b = 3$, $c = 1$, $d = 0$, find the numerical value of

1. $6a + 5b - 8c + 9d$.

2. $3a - 4b + 6c + 5d$.

3. $6ab - 3cd + 2da - 5cb + 2db$.

4. $abc + bcd + cda + dab$.

5. $3abc - 2bcd + 2cda - 4dab$.

6. $2bc + 3cd - 4da + 5ab$.

7. $3bcd + 5cda - 7dab + abc$.

8. $a^2 + b^2 + c^2 + d^2$.

9. $2a^2 + 3b^3 - 4c^4$.

10. $a^4 + b^4 - c^4$.

If $a = 1$, $b = 2$, $c = 3$, $d = 0$, find the numerical value of

11. $a^3 + b^3 + c^3 + d^3$.

12. $\frac{1}{2}bc^3 - a^3 - b^3 - \frac{3}{4}ab^3c$.

13. $3abc - b^2c - 6a^3$.

14. $2a^2 + 2b^2 + 2c^2 + 2d^2 - 2bc - 2cd - 2da - 2ab$.

15. $a^2 + 2b^2 + 2c^2 + d^2 + 2ab + 2bc + \frac{2}{7}cd$.

16. $2c^2 + 2a^2 + 2b^2 - 4cb + 6abcd$.

17. $13a^2 + \frac{11}{9}c^4 + 20ab - 16ac - 16bc$.

18. $6ab - \frac{4}{3}ac^2 - 2a + \frac{1}{8}b^4 - 3d + \frac{4}{9}c^3$.

19. $125b^4c - 9d^5 + 3abc^2d$.

If $a = 8$, $b = 6$, $c = 1$, $x = 9$, $y = 4$, find the value of

20. $\frac{5}{3}a - \frac{1}{9}b^3 + \frac{7}{8}y^2$.

21. $\frac{5}{27}ax - \dfrac{32}{y^2} - \dfrac{6a}{cxy}$.

22. $\dfrac{3a^2b}{cxy^2} - \dfrac{5y}{a}$.

23. $\sqrt[3]{\left(\dfrac{6cy^4}{x^2}\right)} + 2\sqrt{\left(\dfrac{3a^3}{4b^3}\right)}$.

24. $\sqrt[3]{(bxy)} - \frac{1}{8}b^2 + \dfrac{8x^2}{b^2y}$.

25. $\frac{3}{4}ac - \sqrt{\left(\dfrac{b^2}{9y}\right)} - \sqrt[3]{\left(\dfrac{by}{x^2}\right)}$.

26. $\dfrac{5b^2y^3}{12a^2x} - \sqrt[3]{\left(\dfrac{ax^4}{b^2y^2}\right)} + \sqrt{\left(\dfrac{ab^3}{3x}\right)}$.

CHAPTER II.

Negative Quantities. Addition of Like Terms.

19. In his arithmetical work the student has been accustomed to deal with numerical quantities connected by the signs + and – ; and in finding the value of an expression such as $1\frac{3}{4} + 7\frac{2}{3} - 3\frac{1}{8} + 6 - 4\frac{1}{5}$ he understands that the quantities to which the sign + is prefixed are *additive*, and those to which the sign – is prefixed are *subtractive*, while the first quantity, $1\frac{3}{4}$, to which no sign is prefixed, is counted among the additive terms. The same notions prevail in Algebra; thus in using the expression $7a + 3b - 4c - 2d$ we understand the symbols $7a$ and $3b$ to be additive, while $4c$ and $2d$ are subtractive.

20. But in Arithmetic the sum of the additive terms is always greater than the sum of the subtractive terms; and if the reverse were the case, the result would have no arithmetical meaning. In Algebra, however, not only may the sum of the subtractive terms exceed that of the additive, but a subtractive term may stand alone, and yet have a meaning quite intelligible.

Hence all algebraic quantities may be divided into **positive quantities** and **negative quantities**, according as they are expressed with the sign + or the sign – ; and this is quite irrespective of any actual process of addition and subtraction.

This idea may be made clearer by one or two simple illustrations.

(i) Suppose a man were to gain \$100 and then lose \$70, his total *gain* would be \$30. But if he first gains \$70 and then loses \$100 the result of his trading is a *loss* of \$30.

The corresponding algebraic statements would be

$$\$100 - \$70 \ \ = + \$30,$$
$$\$70 \ \ - \$100 = - \$30,$$

and the negative quantity in the second case is interpreted as a *debt*, that is, a sum of money opposite in character to the positive quantity, or *gain*, in the first case; in fact it may be said to possess a subtractive quality which would produce its effect on other transactions, or perhaps wholly counterbalance a sum gained.

(ii) Suppose a man starting from a given point were to walk along a straight road 100 yards forwards and then 70 yards backwards, his distance from the starting-point would be 30 yards. But if he first walks 70 yards forwards and then 100 yards backwards his distance from the starting-point would be 30 yards, but *on the opposite side of it.* As before we have

$$100 \text{ yards} - \ \ 70 \text{ yards} = + 30 \text{ yards},$$
$$70 \text{ yards} - 100 \text{ yards} = - 30 \text{ yards}.$$

In each of these cases the man's *absolute distance* from the starting-point is the same; but by taking the positive and negative signs into account, we see that -30 is a distance from the starting-point *equal in magnitude but opposite in direction* to the distance represented by $+30$. Thus the negative sign may here be taken as indicating *a reversal of direction.*

Many other illustrations might be chosen; but it will be sufficient here to remind the student that a subtractive quantity is always opposite in character to an additive quantity of equal *absolute value.*

NOTE. Absolute value is the value taken independently of the signs $+$ and $-$.

21. DEFINITION. When any number of quantities are connected by the signs $+$ and $-$, the resulting expression is called their **algebraic sum.** Thus $11a - 27a + 13b - 5b$ is an algebraic sum. This expression, however, is not, as will be shown, in its simplest form.

22. Addition is the process of finding in *simplest form* the algebraic sum of any number of quantities.

23. Like terms, or **similar terms,** do not differ, or differ only in their numerical coefficients. Other terms are called **unlike,** or **dissimilar.** Thus $3a$, $7a$; $5a^2b$, $2a^2b$; $3a^3b^2$, $-4a^3b^2$ are pairs of like terms; and $4a$, $3b$; $7a^2$, $9a^2b$ are pairs of unlike terms.

ADDITION OF LIKE TERMS.

Rule I. *The sum of a number of like terms is a like term.*

Rule II. *If all the terms are positive, add the coefficients.*

Ex. Find the value of $8a + 5a$.

Here we have to increase 8 like things by 5 like things of the same kind, and the aggregate is 13 of such things;

for instance, 8 lbs. + 5 lbs. = 13 lbs.

Hence also, $8a + 5a = 13a$.

Similarly, $8a + 5a + a + 2a + 6a = 22a$.

Rule III. *If all the terms are negative, add the coefficients numerically and prefix the minus sign to the sum.*

Ex. To find the sum of $-3x$, $-5x$, $-7x$, $-x$.

Here the word *sum* indicates the aggregate of 4 subtractive quantities of like character. In other words, we have to *take away* successively 3, 5, 7, 1 like things, and the result is the same as taking away $3 + 5 + 7 + 1$ such things in the aggregate.

Thus the sum of $-3x$, $-5x$, $-7x$, $-x$, is $-16x$.

Rule IV. *If the terms are not all of the same sign, add together separately the coefficients of all the positive terms and the coefficients of all the negative terms; the difference of these two results, preceded by the sign of the greater, will give the coefficient of the sum required.*

Ex. 1. The sum of $17x$ and $-8x$ is $9x$, for the difference of 17 and 8 is 9, and the greater is positive.

Ex. 2. To find the sum of $8a$, $-9a$, $-a$, $3a$, $4a$, $-11a$, a.

The sum of the coefficients of the positive terms is 16.

The sum of the coefficients of the negative terms is 21.

The difference of these is 5, and the sign of the greater is negative; hence the required sum is $-5a$.

We need not, however, adhere strictly to this rule, for the terms may be added or subtracted in the order we find most convenient.

This process is called **collecting terms.**

Ex. 3. Find the sum of $\frac{2}{3}a$, $3a$, $-\frac{1}{6}a$, $-2a$.

The sum $= 3\frac{2}{3}a - 2\frac{1}{6}a = 1\frac{1}{2}a = \frac{3}{2}a.$

Note. The sum of two quantities numerically equal but with opposite signs is zero. The sum of $5a$ and $-5a$ is 0.

EXAMPLES II.

Find the sum of

1. $5a$, $7a$, $11a$, a, $23a$.
2. $4x$, x, $3x$, $7x$, $9x$.
3. $7b$, $10b$, $11b$, $9b$, $2b$.
4. $6c$, $8c$, $2c$, $15c$, $19c$, $100c$, c.
5. $-3x$, $-5x$, $-11x$, $-7x$.
6. $-5b$, $-6b$, $-11b$, $-18b$.
7. $-3y$, $-7y$, $-y$, $-2y$, $-4y$.
8. $-c$, $-2c$, $-50c$, $-13c$.
9. $-11b$, $-5b$, $-3b$, $-b$.
10. $5x$, $-x$, $-3x$, $2x$, $-x$.
11. $26y$, $-11y$, $-15y$, y, $-3y$, $2y$.
12. $5f$, $-9f$, $-3f$, $21f$, $-30f$.
13. $2s$, $-3s$, s, $-s$, $-5s$, $5s$.
14. $7y$, $-11y$, $16y$, $-3y$, $-2y$.
15. $5x$, $-7x$, $-2x$, $7x$, $2x$, $-5x$.
16. $7ab$, $-3ab$, $-5ab$, $2ab$, ab.

Find the value of

17. $-9x^2+11x^2+3x^2-4x^2$.
18. $3a^2x-18a^2x+a^2x$.
19. $3a^3-7a^3-8a^3+2a^3-11a^3$.
20. $4x^3-5x^3-8x^3-7x^3$.
21. $4a^2b^2-a^2b^2-7a^2b^2+5a^2b^2-a^2b^2$.
22. $-9x^4-4x^4-12x^4+13x^4-7x^4$.
23. $7abcd-11abcd-41abcd+2abcd$.
24. $\frac{1}{2}x-\frac{1}{3}x+x+\frac{2}{3}x$.
25. $\frac{3}{2}a+\frac{3}{5}a-\frac{1}{2}a$.
26. $-5b+\frac{1}{4}b-\frac{3}{2}b+2b-\frac{1}{2}b+\frac{7}{4}b$.
27. $-\frac{5}{3}x^2-2x^2-\frac{2}{3}x^2+x^2+\frac{1}{2}x^2+\frac{11}{6}x^2$.
28. $-ab-\frac{1}{2}ab-\frac{1}{3}ab-\frac{1}{4}ab-\frac{1}{6}ab+ab+\frac{5}{12}ab$.
29. $\frac{2}{3}x-\frac{3}{4}x+\frac{5}{6}x-2x+\frac{11}{6}x-\frac{1}{3}x+x$.
30. $-\frac{5}{3}x^2-\frac{3}{4}x^2-\frac{4}{3}x^2-\frac{1}{4}x^2-x^2$.

CHAPTER III.

Simple Brackets. Addition. Subtraction.

24. When a number of arithmetical quantities are connected by the signs $+$ and $-$, the value of the result is the same in whatever order the terms are taken. This also holds in the case of algebraic quantities.

Thus $a - b + c$ is equivalent to $a + c - b$, for in the first of the two expressions b is taken from a, and c added to the result; in the second c is added to a, and b taken from the result. Similar reasoning applies to all algebraic expressions. Hence we may write the terms of an expression in any order we please.

Thus it appears that the expression $a - b$ may be written in the equivalent form $-b + a$.

To illustrate this we may suppose, as in Art. 20, that a represents a gain of a dollars, and $-b$ a loss of b dollars: it is clearly immaterial whether the gain precedes the loss, or the loss precedes the gain.

25. Brackets ( ) are used, as in Arithmetic, to indicate that the terms enclosed within them are to be considered as one quantity. The full use of brackets will be considered in Chap. VI.; here we shall deal only with the simpler cases.

$8 + (13 + 5)$ means that 13 and 5 are to be added and their sum added to 8. It is clear that 13 and 5 may be added separately or together without altering the result.

Thus $$8 + (13 + 5) = 8 + 13 + 5 = 26.$$

Similarly $a + (b + c)$ means that the sum of b and c is to be added to a.

Thus $$a + (b + c) = a + b + c.$$

$8 + (13 - 5)$ means that to 8 we are to add the excess of 13 over 5; now if we add 13 to 8 we have added 5 too much, and must therefore take 5 from the result.

Thus $$8 + (13 - 5) = 8 + 13 - 5 = 16.$$

Similarly $a + (b - c)$ means that to a we are to add b, diminished by c.

Thus $$a + (b - c) = a + b - c \quad . \quad . \quad . \quad . \quad (1).$$

In like manner,

$$a + b - c + (d - e - f) = a + b - c + d - e - f. \quad (2).$$

Conversely,

$$a + b - c + d - e - f = a + b - c + (d - e - f). \quad (3).$$

Again, $a - b + c = a + c - b,$ [Art. 24.]

$= \text{the sum of } a \text{ and } c - b,$

$= \text{the sum of } a \text{ and } -b + c,$ [Art. 24.]

therefore $$a - b + c = a + (-b + c) \quad . \quad . \quad . \quad . \quad (4).$$

By considering the results (1), (2), (3), (4), we are led to the following rule:

Rule. *When an expression within brackets is preceded by the sign +, the brackets can be removed without making any change in the expression.*

Conversely: *Any part of an expression may be enclosed within brackets and the sign + prefixed, the sign of every term within the brackets remaining unaltered.*

Thus the expression $a - b + c - d + e$ may be written in any of the following ways,

$$a + (-b + c - d + e)$$
$$a - b + (c - d + e),$$
$$a - b + c + (-d + e).$$

26. The expression $a - (b + c)$ means that from a we are to take the sum of b and c. The result will be the same whether b and c are subtracted separately or in one sum.

Thus $$a - (b + c) = a - b - c.$$

Again, $a - (b - c)$ means that from a we are to subtract the excess of b over c. If from a we take b we get $a - b$; but by so doing we shall have taken away c too much, and must therefore add c to $a - b$. Thus

$$a - (b - c) = a - b + c.$$

In like manner, $a - b - (c - d - e) = a - b - c + d + e$.

Accordingly the following rule may be enunciated:

Rule. *When an expression within brackets is preceded by the sign $-$, the brackets may be removed if the sign of every term within the brackets be changed.*

Conversely: *Any part of an expression may be enclosed within brackets and the sign $-$ prefixed, provided the sign of every term within the brackets be changed.*

Thus the expression $a - b + c + d - e$ may be written in any of the following ways,

$$a - (+b - c - d + e),$$
$$a - b - (-c - d + e),$$
$$a - b + c - (-d + e).$$

We have now established the following results:

I. *Additions and subtractions may be made in any order.*

Thus $a + b - c + d - e - f = a - c + b + d - f - e$
$$= a - c - f + d + b - e.$$

This is known as the **Commutative Law for Addition and Subtraction.**

II. *The terms of an expression may be grouped in any manner.*

Thus $a + b - c + d - e - f = (a + b) - c + (d - e) - f$
$$= a + (b - c) + (d - e) - f = a + b - (c - d) - (e + f).$$

This is known as the **Associative Law for Addition and Subtraction.**

ADDITION OF UNLIKE TERMS.

27. When two or more *like* terms are to be added together we have seen that they may be collected and the result expressed as a single like term. If, however, the terms are *unlike*, they cannot be collected. Thus in finding the sum of two unlike quantities a and b, all that can be done is to connect them by the sign of addition and leave the result in the form $a + b$.

Also, by the rules for removing brackets, $a+(-b)=a-b$; that is, the algebraic sum of a and $-b$ is written in the form $a - b$.

28. It will be observed that in Algebra the word *sum* is used in a wider sense than in Arithmetic. Thus, in the language of Arithmetic, $a - b$ signifies that b is to be subtracted from a, and bears that meaning only; but in Algebra it also means the sum of the two quantities a and $-b$ without any regard to the relative magnitudes of a and b.

Ex. 1. Find the sum of $3a - 5b + 2c$; $2a + 3b - d$; $-4a + 2b$.

$$\begin{aligned}\text{The sum } &= (3a - 5b + 2c) + (2a + 3b - d) + (-4a + 2b)\\ &= 3a - 5b + 2c + 2a + 3b - d - 4a + 2b\\ &= 3a + 2a - 4a - 5b + 3b + 2b + 2c - d\\ &= a + 2c - d,\end{aligned}$$

by collecting like terms.

The addition is more conveniently effected by the following rule:

Rule. *Arrange the expressions in lines so that the like terms may be in the same vertical columns: then add each column, beginning with that on the left.*

$$\begin{array}{rrrr} 3a & -5b & +2c & \\ 2a & +3b & & -d \\ -4a & +2b & & \\ \hline a & & +2c & -d \end{array}$$

The algebraic sum of the terms in the first column is a, that of the terms in the second column is zero. The single terms in the third and fourth columns are brought down without change.

Ex. 2. Add together $-5ab + 6bc - 7ac$; $8ab + 3ac - 2ad$; $-2ab + 4ac + 5ad$; $bc - 3ab + 4ad$.

$$\begin{array}{rrrr} -5ab & +6bc & -7ac & \\ 8ab & & +3ac & -2ad \\ -2ab & & +4ac & +5ad \\ -3ab & +bc & & +4ad \\ \hline -2ab & +7bc & & +7ad \end{array}$$

Here we first rearrange the expressions so that like terms are in the same vertical columns, and then add up each column separately.

EXAMPLES III. a.

Find the sum of

1. $a + 2b - 3c$; $-3a + b + 2c$; $2a - 3b + c$.
2. $3a + 2b - c$; $-a + 3b + 2c$; $2a - b + 3c$.
3. $-3x + 2y + z$; $x - 3y + 2z$; $2x + y - 3z$.
4. $-x + 2y + 3z$; $3x - y + 2z$; $2x + 3y - z$.
5. $4a + 3b + 5c$; $-2a + 3b - 8c$; $a - b + c$.
6. $-15a - 19b - 18c$; $14a + 15b + 8c$; $a + 5b + 9c$.
7. $25a - 15b + c$; $13a - 10b + 4c$; $a + 20b - c$.
8. $-16a - 10b + 5c$; $10a + 5b + c$; $6a + 5b - c$.
9. $5ax - 7by + cz$; $ax + 2by - cz$; $-3ax + 2by + 3cz$.
10. $20p + q - r$; $p - 20q + r$; $p + q - 20r$.

Add together the following expressions

11. $-5ab + 6bc - 7ca$; $8ab - 4bc + 3ca$; $-2ab - 2bc + 4ca$.
12. $15ab - 27bc - 6ca$; $14ab - 18bc + 10ca$; $-49ab + 45bc - 3ca$.
13. $5ab + bc - 3ca$; $ab - bc + ca$; $-ab + bc + 2ca$.
14. $pq + qr - rp$; $-pq + qr + rp$; $pq - qr + rp$.
15. $x + y + z$; $2x + 3y - 2z$; $3x - 4y + z$.
16. $2a - 3b + c$; $15a - 21b - 8c$; $3a + 24b + 7c$.
17. $4xy - 9yz + 2zx$; $-25xy + 24yz - zx$; $23xy - 15yz + zx$.
18. $17ab - 13bc + 8ca$; $-5ab + 9bc - 7ca$; $2ab - 7bc - ca$.
19. $47x - 63y + z$; $-25x + 15y - 3z$; $-22x + 48y + 15z$.
20. $23a - 17b - 2c$; $-9a + 15b + 7c$; $-13a + 3b - 4c$.

DIMENSION, DEGREE, ASCENDING AND DESCENDING POWERS.

29. Each of the letters composing a term is called a **dimension** of the term, and the number of letters involved is called the **degree** of the term. Thus the product abc is said to be *of three dimensions*, or *of the third degree;* and ax^4 is said to be *of five dimensions*, or *of the fifth degree.*

A numerical coefficient is not counted. Thus $8a^2b^5$ and a^2b^5 are each *of seven dimensions*, or *of the seventh degree.*

But it is sometimes useful to speak of the dimensions of an expression with regard to any one of the letters it involves. For instance, the expression $8a^3b^4c$, which is of eight dimensions, may be said to be of three dimensions in a, of four dimensions in b, and of one dimension in c.

30. A compound expression is said to be **homogeneous** when all its terms are of the same degree. Thus $8a^6 - a^4b^2 + 9ab^5$ is a *homogeneous expression of six dimensions*, or *of the sixth degree.*

31. Different powers of the same letter are **unlike terms**; thus the result of adding together $2x^3$ and $3x^2$ cannot be expressed by a single term, but must be left in the form $2x^3 + 3x^2$.

Similarly, the algebraic sum of $5a^2b^2 - 3ab^3$ and $-b^4$ is $5a^2b^2 - 3ab^3 - b^4$. This expression is in its simplest form and cannot be abridged.

32. In adding together several algebraic expressions containing terms with different powers of the same letter, it will be found convenient to arrange all the expressions in **descending** or **ascending powers** of that letter. This will be made clear by the following examples.

Ex. 1. Add together $3x^3 + 7 + 6x - 5x^2$; $2x^2 - 8 - 9x$; $4x - 2x^3 + 3x^2$; $3x^3 - 9x - x^2$; $x - x^2 - x^3 + 4$.

$$\begin{array}{rrrr} 3x^3 & -5x^2 & +6x & +7 \\ & 2x^2 & -9x & -8 \\ -2x^3 & +3x^2 & +4x & \\ 3x^3 & -x^2 & -9x & \\ -x^3 & -x^2 & +x & +4 \\ \hline 3x^3 & -2x^2 & -7x & +3 \end{array}$$

In writing the first expression we put in the first term the highest power of x, in the second term the next highest power, and so on till the last term, in which x does not appear. The other expressions are arranged in the same way, so that in each column we have *like powers of the same letter*. The result is in *descending* powers of x.

Ex. 2. Add together

$3ab^2 - 2b^3 + a^3$; $5a^2b - ab^2 - 3a^3$; $8a^3 + 5b^3$; $9a^2b - 2a^3 + ab^2$.

$$\begin{array}{rrrr} -2b^3 & +3ab^2 & & +a^3 \\ & -ab^2 & +5a^2b & -3a^3 \\ 5b^3 & & & +8a^3 \\ & ab^2 & +9a^2b & -2a^3 \\ \hline 3b^3 & +3ab^2 & +14a^2b & +4a^3 \end{array}$$

Here each expression is arranged according to *descending* powers of b, and *ascending* powers of a.

EXAMPLES III. b.

Find the sum of the following expressions :

1. $2ab + 3ac + 6abc$; $-5ab + 2bc - 5abc$; $3ab - 2bc - 3ac$.
2. $2x^2 - 2xy + 3y^2$; $4y^2 + 5xy - 2x^2$; $x^2 - 2xy - 6y^2$.
3. $3a^2 - 7ab - 4b^2$; $-6a^2 + 9ab - 3b^2$; $4a^2 + ab + 5b^2$.
4. $x^2 + xy - y^2$; $-z^2 + yz + y^2$; $-x^2 + xz + z^2$.
5. $-x^2 - 3xy + 3y^2$; $3x^2 + 4xy - 5y^2$; $x^2 + xy + y^2$.
6. $x^3 - x^2 + x - 1$; $2x^2 - 2x + 2$; $-3x^3 + 5x + 1$.
7. $2x^3 - x^2 - x$; $4x^3 + 8x^2 + 7x$; $-6x^3 - 6x^2 + x$.
8. $9x^2 - 7x + 5$; $-14x^2 + 15x - 6$; $20x^2 - 40x - 17$.
9. $10x^3 + 5x + 8$; $3x^3 - 4x^2 - 6$; $2x^3 - 2x - 3$.
10. $a^3 - ab + bc$; $ab + b^3 - ac$; $ac - bc + c^3$.
11. $5a^3 - 3c^3 + d^3$; $b^3 - 2a^3 + 3d^3$; $4c^3 - 2a^3 - 3d^3$.
12. $6x^3 - 2x + 1$; $2x^3 + x + 6$; $x^2 - 7x^3 + 2x - 4$.
13. $a^3 - a^2 + 3a$; $3a^3 + 4a^2 + 8a$; $5a^3 - 6a^2 - 11a$.
14. $x^2 + y^2 - 2xy$; $2z^2 - 3y^2 - 4yz$; $2x^2 - 2z^2 - 3xz$.
15. $x^3 - 2y^3 + x$; $y^3 - 2x^3 + y$; $x^2 + 2y^2 - x + y^3$.
16. $x^3 + 3x^2y + 3xy^2$; $-3x^2y - 6xy^2 - x^3$; $3x^2y + 4xy^2$.
17. $a^3 + 5ab^2 + b^3$; $b^3 - 10ab^2 - a^3$; $5ab^2 - 2b^3 + 2a^2b$.
18. $x^5 - 4x^4y - 5x^3y^3$; $3x^4y + 2x^3y^3 - 6xy^4$; $3x^3y^3 + 6xy^4 - y^5$.
19. $a^3 - 4a^2b + 6abc$; $a^2b - 10abc + c^3$; $b^3 + 3a^2b + abc$.
20. $x^3 - 4x^2y + 6xy^2$; $2x^2y - 3xy^2 + 2y^3$; $y^3 + 3x^2y + 4xy^2$.

Add together the following expressions :

21. $\frac{1}{2}a - \frac{1}{3}b$; $-a + \frac{2}{3}b$; $\frac{3}{4}a - b$.
22. $-\frac{1}{3}a - \frac{1}{4}b$; $-\frac{2}{3}a + \frac{3}{4}b$; $-2a - b$.
23. $-2a + \frac{5}{2}c$; $-\frac{1}{3}a - 2b$; $\frac{8}{3}b - 3c$.
24. $-\frac{13}{8}a - \frac{11}{4}c$; $2a - 3b$; $\frac{11}{5}b - c$.
25. $\frac{2}{3}x^2 + \frac{1}{3}xy - \frac{1}{4}y^2$; $-x^2 - \frac{2}{3}xy + 2y^2$; $\frac{2}{3}x^2 - xy - \frac{5}{4}y^2$.
26. $3a^2 - \frac{2}{5}ab - \frac{1}{2}b^2$; $-\frac{3}{2}a^2 + 2ab - \frac{2}{3}b^2$; $-\frac{2}{3}a^2 - ab + b^2$.
27. $\frac{5}{8}x^2 - \frac{1}{3}xy + \frac{3}{10}y^2$; $-\frac{3}{4}x^2 + \frac{14}{15}xy - y^2$; $\frac{1}{2}x^2 - xy + \frac{1}{5}y^2$.
28. $-\frac{3}{4}x^3 + 5ax^2 - \frac{5}{8}a^2x$; $x^3 - \frac{37}{8}ax^2 + \frac{1}{2}a^2x$; $-\frac{1}{2}x^3 + \frac{3}{4}a^2x$.
29. $\frac{3}{8}x^2 - \frac{5}{3}xy - 7y^2$; $\frac{2}{3}xy + \frac{18}{5}y^2$; $-\frac{5}{8}x^2 + 4y^2$.
30. $\frac{1}{2}a^3 - 2a^2b - \frac{3}{2}b^3$; $\frac{3}{2}a^2b - \frac{3}{4}ab^2 + 2b^3$; $-\frac{3}{2}a^3 + ab^2 + \frac{1}{2}b^3$.

SUBTRACTION.

33. Subtraction is the inverse of Addition. The simplest cases have been considered under the head of addition of *like* terms, of which some are negative. [Art. 23.]

Thus
$$5a - 3a = 2a,$$
$$3a - 7a = -4a,$$
$$-3a - 6a = -9a.$$

Also, by the rule for removing brackets [Art. 26],
$$3a - (-8a) = 3a + 8a$$
$$= 11a,$$
and
$$-3a - (-8a) = -3a + 8a$$
$$= 5a.$$

SUBTRACTION OF UNLIKE TERMS.

34. The method is shown in the following example:

Ex. Subtract $3a - 2b - c$ from $4a - 3b + 5c$.

The result of subtraction $= 4a - 3b + 5c - (3a - 2b - c)$
$$= 4a - 3b + 5c - 3a + 2b + c$$
$$= 4a - 3a - 3b + 2b + 5c + c$$
$$= a - b + 6c.$$

It is, however, more convenient to arrange the work as follows, *the signs of all the terms in the lower line being changed.*

$$\begin{array}{r} 4a - 3b + 5c \\ -3a + 2b + c \\ \hline a \, - b + 6c \end{array}$$
by addition

Rule. *Change the sign of every term in the expression to be subtracted, and add it to the other expression.*

Note. It is not necessary that in the expression to be subtracted the signs should be *actually* changed; the operation of changing signs ought to be performed mentally.

Ex. 1. From $5x^2 + xy - 3y^2$ take $2x^2 + 8xy - 7y^2$.

$$\begin{array}{r} 5x^2 + xy - 3y^2 \\ 2x^2 + 8xy - 7y^2 \\ \hline 3x^2 - 7xy + 4y^2. \end{array}$$

Ex. 2. Subtract $3x^2 - 2x$ from $1 - x^3$.

Terms containing different powers of the same letter being *unlike* must stand in different columns.

$$\begin{array}{rrrr} -x^3 & & & +1 \\ & 3x^2 & -2x & \\ \hline -x^3 & -3x^2 & +2x & +1 \end{array}$$

The rearrangement of terms in the first line is not *necessary*, but it is convenient, because it gives the result of subtraction in descending powers of x.

EXAMPLES III. c.

Subtract

1. $4a - 3b + c$ from $2a - 3b - c$.
2. $a - 3b + 5c$ from $4a - 8b + c$.
3. $2x - 8y + z$ from $15x + 10y - 18z$.
4. $15a - 27b + 8c$ from $10a + 3b + 4c$.
5. $-10x - 14y + 15z$ from $x - y - z$.
6. $-11ab + 6cd$ from $-10bc + ab - 4cd$.
7. $4a - 3b + 15c$ from $25a - 16b - 18c$.
8. $-16x - 18y - 15z$ from $-5x + 8y + 7z$.
9. $ab + cd - ac - bd$ from $ab + cd + ac + bd$.
10. $-ab + cd - ac + bd$ from $ab - cd + ac - bd$.

From

11. $3ab + 5cd - 4ac - 6bd$ take $3ab + 6cd - 3ac - 5bd$.
12. $yz - xz + xy$ take $-xy + yz - xz$.
13. $-2x^3 - x^2 - 3x + 2$ take $x^3 - x + 1$.
14. $-8x^2y + 15xy^2 + 10xyz$ take $4x^2y - 6xy^2 - 5xyz$.
15. $\frac{1}{2}a - b + \frac{1}{3}c$ take $\frac{1}{3}a + \frac{1}{2}b - \frac{1}{2}c$.
16. $\frac{3}{4}x + y - z$ take $\frac{1}{2}x - \frac{1}{2}y - \frac{1}{3}z$.
17. $-a - 3b$ take $\frac{3}{2}a + \frac{1}{3}b - \frac{1}{2}c$.
18. $\frac{1}{2}x - \frac{3}{7}y + \frac{1}{10}z$ take $-\frac{1}{2}x + \frac{4}{7}y - \frac{1}{10}z$.
19. $-\frac{2}{3}x - \frac{3}{5}y - 5z$ take $\frac{2}{3}x - \frac{3}{5}y - \frac{11}{3}z$.
20. $-\frac{1}{2}x + \frac{2}{3}y - \frac{1}{6}$ take $\frac{1}{3}x - \frac{3}{2}y - \frac{1}{6}$.

EXAMPLES III. d.

From

1. $3xy - 5yz + 8xz$ take $-4xy + 2yz - 10xz$.
2. $-8x^2y^2 + 15x^3y + 13xy^3$ take $4x^2y^2 + 7x^3y - 8xy^3$.
3. $-8 + 6ab + a^2b^2$ take $4 - 3ab - 5a^2b^2$.
4. $a^2bc + b^2ca + c^2ab$ take $3a^2bc - 5b^2ca - 4c^2ab$.
5. $-7a^2b + 8ab^2 + cd$ take $5a^2b - 7ab^2 + 6cd$.
6. $-8x^2y + 5xy^2 - x^2y^2$ take $8x^2y - 5xy^2 + x^2y^2$.
7. $10a^2b^2 + 15ab^2 + 8a^2b$ take $-10a^2b^2 + 15ab^2 - 8a^2b$.
8. $4x^2 - 3x + 2$ take $-5x^2 + 6x - 7$.
9. $x^3 + 11x^2 + 4$ take $8x^2 - 5x - 3$.
10. $-8a^2x^2 + 5x^2 + 15$ take $9a^2x^2 - 8x^2 - 5$.

Subtract

11. $x^3 - x^2 + x + 1$ from $x^3 + x^2 - x + 1$.
12. $3xy^2 - 3x^2y + x^3 - y^3$ from $x^3 + 3x^2y + 3xy^2 + y^3$.
13. $b^3 + c^3 - 2abc$ from $a^3 + b^3 - 3abc$.
14. $7xy^2 - y^3 - 3x^2y + 5x^3$ from $8x^3 + 7x^2y - 3xy^2 - y^3$.
15. $x^4 + 5 + x - 3x^3$ from $5x^4 - 8x^3 - 2x^2 + 7$.
16. $a^3 + b^3 + c^3 - 3abc$ from $7abc - 3a^3 + 5b^3 - c^3$.
17. $1 - x + x^5 - x^4 - x^3$ from $x^4 - 1 + x - x^2$.
18. $7a^4 - 8a^2 + 3a^5 + a$ from $a^2 - 5a^3 - 7 + 7a^5$.
19. $10a^2b + 8ab^2 - 8a^3b^3 - b^4$ from $5a^2b - 6ab^2 - 7a^3b^3$.
20. $a^3 - b^3 + 8ab^2 - 7a^2b$ from $-8ab^2 + 15a^2b + b^3$.

From

21. $\frac{1}{2}x^2 - \frac{1}{3}xy - \frac{3}{2}y^2$ take $-\frac{3}{2}x^2 + xy - y^2$.
22. $\frac{2}{3}a^2 - \frac{5}{2}a - 1$ take $-\frac{2}{3}a^2 + a - \frac{1}{2}$.
23. $\frac{1}{3}x^2 - \frac{1}{2}x + \frac{1}{6}$ take $\frac{1}{3}x - 1 + \frac{1}{2}x^2$.
24. $\frac{3}{8}x^2 - \frac{2}{3}ax$ take $\frac{1}{3} - \frac{1}{4}x^2 - \frac{5}{6}ax$.
25. $\frac{3}{4}x^3 - \frac{1}{3}xy^2 - y^2$ take $\frac{1}{2}x^2y - \frac{5}{6}y^2 - \frac{1}{3}xy^2$.
26. $\frac{1}{8}a^3 - 2ax^2 - \frac{1}{3}a^2x$ take $\frac{1}{3}a^2x + \frac{1}{4}a^3 - \frac{3}{2}ax^2$.

35. We shall close this chapter with an exercise containing miscellaneous examples of Addition and Subtraction.

MISCELLANEOUS EXAMPLES I.

1. To the sum of $2a - 3b - 2c$ and $2b - a + 7c$ add the sum of $a - 4c + 7b$ and $c - 6b$.

2. From $5x^3 + 3x - 1$ take the sum of $2x - 5 + 7x^2$ and $3x^2 + 4 - 2x^3 + x$.

3. Subtract $3a - 7a^3 + 5a^2$ from the sum of $2 + 8a^2 - a^3$ and $2a^3 - 3a^2 + a - 2$.

4. Subtract $5x^2 + 3x - 1$ from $2x^3$, and add the result to $3x^2 + 3x - 1$.

5. Add the sum of $2y - 3y^2$ and $1 - 5y^3$ to the remainder when $1 - 2y^2 + y$ is subtracted from $5y^3$.

6. Take $x^2 - y^2$ from $3xy - 4y^2$, and add the remainder to the sum of $4xy - x^2 - 3y^2$ and $2x^2 + 6y^2$.

7. Find the sum of $5a - 7b + c$ and $3b - 9a$, and subtract the result from $c - 4b$.

8. Add together $3x^2 - 7x + 5$ and $2x^3 + 5x - 3$, and diminish the result by $3x^2 + 2$.

9. What expression must be added to $5x^2 - 7x + 2$ to produce $7x^2 - 1$?

10. What expression must be added to $4x^3 - 3x^2 + 2$ to produce $4x^3 + 7x - 6$?

11. What expression must be subtracted from $3a - 5b + c$ so as to leave $2a - 4b + c$?

12. What expression must be subtracted from $9x^2 + 11x - 5$ so as to leave $6x^2 - 17x + 3$?

13. From what expression must $11a^2 - 5ab - 7bc$ be subtracted so as to give for remainder $5a^2 + 7ab + 7bc$?

14. From what expression must $3ab + 5bc - 6ca$ be subtracted so as to leave a remainder $6ca - 5bc$?

15. To what expression must $7x^3 - 6x^2 - 5x$ be added so as to make $9x^3 - 6x - 7x^2$?

16. To what expression must $5ab - 11bc - 7ca$ be added so as to produce zero?

17. If $3x^2 - 7x + 2$ be subtracted from zero, what will be the result?

18. Subtract $3x^3 - 7x + 1$ from $2x^2 - 5x - 3$, then subtract the difference from zero, and add this last result to $2x^2 - 2x^3 - 4$.

19. Subtract $3x^2 - 5x + 1$ from unity, and add $5x^2 - 6x$ to the result.

CHAPTER IV.

MULTIPLICATION.

36. Multiplication in its primary sense signifies repeated addition.

Thus $$3 \times 4 = 3 \text{ taken } 4 \text{ times}$$
$$= 3 + 3 + 3 + 3.$$

Here the multiplier contains 4 units, and the number of times we take 3 is the same as the number of units in 4.

Again $$a \times b = a \text{ taken } b \text{ times}$$
$$= a + a + a + \cdots, \text{ the number of terms being } b.$$

Also $3 \times 4 = 4 \times 3$; and so long as a and b denote positive whole numbers, it is easy to show that $a \times b = b \times a$.

37. When the two quantities to be multiplied together are not positive whole numbers, we may define multiplication as *an operation performed on one quantity which when performed on unity produces the other.* For example, to multiply $\frac{4}{5}$ by $\frac{3}{7}$, we perform on $\frac{4}{5}$ that operation which when performed on unity gives $\frac{3}{7}$; that is, we must divide $\frac{4}{5}$ into 7 equal parts and take 3 of them. Now each part will be equal to $\dfrac{4}{5 \times 7}$, and the result of taking 3 of such parts is expressed by $\dfrac{4 \times 3}{5 \times 7}$.

Hence $$\frac{4}{5} \times \frac{3}{7} = \frac{4 \times 3}{5 \times 7}.$$

Also, by the last article,

$$\frac{4 \times 3}{5 \times 7} = \frac{3 \times 4}{7 \times 5} = \frac{3}{7} \times \frac{4}{5}.$$

$$\therefore \frac{4}{5} \times \frac{3}{7} = \frac{3}{7} \times \frac{4}{5}.$$

The reasoning is clearly general, and we may now say that $a \times b = b \times a$, where a and b are any positive quantities, integral or fractional.

The same is true for any number of quantities, hence *the factors of a product may be taken in any order.* This is the **Commutative Law for Multiplication.**

38. Again, *the factors of a product may be grouped in any way we please.*

Thus $$abcd = a \times b \times c \times d$$
$$= (ab) \times (cd) = a \times (bc) \times d = a \times (bcd).$$

This is the **Associative Law for Multiplication.**

39. Since, by definition, $a^3 = aaa$, and $a^5 = aaaaa$,

$$\therefore a^3 \times a^5 = aaa \times aaaaa = aaaaaaaa = a^8 = a^{3+5};$$

that is, *the index of a letter in the product is the sum of its indices in the factors of the product.* This is the **Index Law for Multiplication.**

Again, $$5a^2 = 5aa, \text{ and } 7a^3 = 7aaa.$$
$$\therefore 5a^2 \times 7a^3 = 5 \times 7 \times aaaaa = 35a^5.$$

When the expressions to be multiplied together contain powers of different letters, a similar method is used.

Ex. $5a^3b^2 \times 8a^2bx^3 = 5aaabb \times 8aabxxx = 40a^5b^3x^3$.

Note. The beginner must be careful to observe that in this process of multiplication *the indices of one letter cannot combine in any way with those of another.* Thus the expression $40a^5b^3x^3$ admits of no further simplification.

40. Rule. *To multiply two simple expressions together, multiply the coefficients together and prefix their product to the product of the different letters, giving to each letter an index equal to the sum of the indices that letter has in the separate factors.*

The rule may be extended to cases where more than two expressions are to be multiplied together.

Ex. 1. Find the product of x^2, x^3, and x^8.

The product $= x^2 \times x^3 \times x^8 = x^{2+3} \times x^8 = x^{2+3+8} = x^{13}$.

The product of three or more expressions is called **the continued product.**

Ex. 2. Find the continued product of $5x^2y^3$, $8y^2z^5$, and $3xz^4$.

The product $= 5x^2y^3 \times 8y^2z^5 \times 3xz^4 = 120x^3y^5z^9$.

MULTIPLICATION OF A COMPOUND EXPRESSION BY A SIMPLE EXPRESSION.

41. By definition,

$$(a+b)m = m+m+m+\cdots \text{ taken } a+b \text{ times}$$
$$=(m+m+m+\cdots \text{ taken } a \text{ times}),$$

together with $\quad (m+m+m+\cdots \text{ taken } b \text{ times})$

$$= am + bm \quad \ldots\ldots\ldots \quad (1).$$

Also $\quad (a-b)m = m+m+m+\cdots \text{ taken } a-b \text{ times}$

$$=(m+m+m+\cdots \text{ taken } a \text{ times}),$$

diminished by $\quad (m+m+m+\cdots \text{ taken } b \text{ times})$

$$= am - bm \quad \ldots\ldots\ldots \quad (2).$$

Similarly, $\quad (a-b+c)m$

$$= am - bm + cm.$$

Hence *the product of a compound expression by a single factor is the algebraic sum of the partial products of each term of the compound expression by that factor.* This is known as the **Distributive Law for Multiplication.**

Ex. $3(2a + 3b - 4c) = 6a + 9b - 12c.$

$$(4x^2 - 7y - 8z^3) \times 3xy^2 = 12x^3y^2 - 21xy^3 - 24xy^2z^3.$$

NOTE. It should be observed that for the present a, b, c, m denote positive whole numbers, and that a is supposed to be greater than b.

EXAMPLES IV. a.

Find value of

1. $5x^2 \times 7x^5$.
2. $4a^3 \times 5a^8$.
3. $7ab \times 8a^3b^2$.
4. $6xy^2 \times 5x^3$.
5. $8a^3b \times b^5$.
6. $2abc \times 3ac^3$.
7. $2a^3b^3 \times 2a^3b^3$.
8. $5a^2b \times 2a$.
9. $4a^2b^3 \times 7a^5$.
10. $5a^4b^3 \times x^2y^2$.
11. $x^3y^3 \times 6a^2x^4$.
12. $abc \times xyz$.
13. $3a^4b^7x^3 \times 5a^3bx$.
14. $4a^3bx \times 7b^2x^4$.
15. $5a^2x \times 8cx$.

Multiply

16. $5x^3y^3$ by $6a^3x^3$.
17. $2x^2y$ by x^5y^7.
18. $3a^3x^4y^7$ by $a^2x^5y^9$.
19. $ab + bc$ by a^3b.
20. $5ab - 7bx$ by $4a^2bx^3$.
21. $5x + 3y$ by $2x^2$.
22. $a^2 + b^2 - c^2$ by a^3b.
23. $bc + ca - ab$ by abc.
24. $5a^2 + 3b^2 - 2c^2$ by $4a^2bc^3$.
25. $5x^2y + xy^2 - 7x^2y^2$ by $3x^3$.

MULTIPLICATION OF COMPOUND EXPRESSIONS.

42. If in Art. 41 we write $c + d$ for m in (1), we have

$$\begin{aligned}(a + b)(c + d) &= a(c + d) + b(c + d)\\ &= (c + d)a + (c + d)b \qquad \text{[Art. 37.]}\\ &= ac + ad + bc + bd.\end{aligned}$$

Again, from (2)

$$\begin{aligned}(a - b)(c + d) &= a(c + d) - b(c + d)\\ &= (c + d)a - (c + d)b\\ &= ac + ad - (bc + bd)\\ &= ac + ad - bc - bd.\end{aligned}$$

Similarly, by writing $c - d$ for m in (1)

$$\begin{aligned}(a + b)(c - d) &= a(c - d) + b(c - d)\\ &= (c - d)a + (c - d)b\\ &= ac - ad + bc - bd.\end{aligned}$$

Also, from (2)

$$\begin{aligned}(a-b)(c-d) &= a(c-d)-b(c-d)\\ &= (c-d)a-(c-d)b\\ &= ac-ad-(bc-bd)\\ &= ac-ad-bc+bd.\end{aligned}$$

If we consider each term on the right-hand side of this last result, and the way in which it arises, we find that

$$\begin{aligned}(+a)\times(+c) &= +ac,\\ (-b)\times(-d) &= +bd,\\ (-b)\times(+c) &= -bc,\\ (+a)\times(-d) &= -ad.\end{aligned}$$

These results enable us to state what is known as the **Rule of Signs** in multiplication.

Rule of Signs. *The product of two terms with like signs is positive; the product of two terms with unlike signs is negative.*

43. The rule of signs, and especially the use of the negative multiplier, will probably present some difficulty to the beginner. Perhaps the following numerical instances may be useful in illustrating the interpretation that may be given to multiplication by a negative quantity.

To multiply 3 by -4 we must do to 3 what is done to unity to obtain -4. Now -4 means that unity is taken 4 times and the result made negative; therefore $3\times(-4)$ implies that 3 is to be taken 4 times and the product made negative.

But 3 taken 4 times gives $+12$.

$$\therefore\ 3\times(-4)=-12.$$

Similarly, -3×-4 indicates that -3 is to be taken 4 times, and the sign changed; the first operation gives -12, and the second $+12$.

Thus $$(-3)\times(-4)=+12.$$

Hence, *multiplication by a negative quantity indicates that we are to proceed just as if the multiplier were positive, and then change the sign of the product.*

44. Note on Arithmetical and Symbolical Algebra.

Arithmetical Algebra is that part of the science which deals solely with symbols and operations arithmetically intelligible. Starting from purely arithmetical definitions, we are enabled to prove certain fundamental laws.

Symbolical Algebra assumes these laws to be true in every case, and thence finds what meaning must be attached to symbols and operations which under unrestricted conditions no longer bear an arithmetical meaning. Thus the results of Arts. 41 and 42 were proved from arithmetical definitions which require the symbols to be positive whole numbers, such that a is greater than b and c is greater than d. By the principles of Symbolical Algebra we assume these results to be universally true when all restrictions are removed, and accept the interpretation to which we are led thereby.

Henceforth we are able to apply the Law of Distribution and the Rule of Signs without any restriction as to the symbols used.

45. To familiarize the beginner with the principles we have just explained we add a few examples in substitutions where some of the symbols denote negative quantities.

Ex. **1.** If $a = -4$, find the value of a^3.

Here $a^3 = (-4)^3 = (-4) \times (-4) \times (-4) = -64.$

Ex. **2.** If $a = -1$, $b = 3$, $c = -2$, find the value of $-3a^4bc^3$.

Here
$$-3a^4bc^3 = -3 \times (-1)^4 \times 3 \times (-2)^3$$
$$= -3 \times 1 \times 3 \times (-8) = 72.$$

EXAMPLES IV. b.

If $a = -2$, $b = 3$, $c = -1$, $x = -5$, $y = 4$, find the value of

1. $3a^2b$.	**6.** $3a^2c$.	**11.** $-4a^2c^4$.	**16.** $4c^5x^3$.
2. $8abc^2$.	**7.** $-b^2c^2$.	**12.** $3c^3x^3$.	**17.** $-5a^2b^2c^2$.
3. $-5c^3$.	**8.** $3a^3c^2$.	**13.** $5a^2x^2$.	**18.** $-7a^3c^3$.
4. $6a^2c^2$.	**9.** $-7a^3bc$.	**14.** $-7c^4xy$.	**19.** $8c^4x^3$.
5. $4c^3y$.	**10.** $-2a^4bx$.	**15.** $-8ax^3$.	**20.** $7a^5c^4$.

If $a=-4$, $b=-3$, $c=-1$, $f=0$, $x=4$, $y=1$, find the value of

21. $3a^2+bx-4cy$.

22. $2ab^2-3bc^2+2fx$.

23. $fa^2-2b^3-cx^3$.

24. $3a^2y^3-5b^2x-2c^3$.

25. $2a^3-3b^3+7cy^4$.

26. $3b^2y^4-4b^2f-6c^4x$.

27. $2\sqrt{(ac)}-3\sqrt{(xy)}+\sqrt{(b^2c^4)}$.

28. $3\sqrt{(acx)}-2\sqrt{(b^2y)}-6\sqrt{(c^2y)}$.

29. $7\sqrt{(a^2x)}-3\sqrt{(b^4c^2)}+5\sqrt{(f^2x)}$.

30. $3c\sqrt{(3bc)}-5\sqrt{(4c^2y^3)}-2cy\sqrt{(3bc^5)}$.

46. The following examples further illustrate the rule of signs and the law of indices.

Ex. 1. Multiply $4a$ by $-3b$.

By the rule of signs the product is negative; also $4a\times 3b=12ab$.

$$\therefore 4a\times(-3b)=-12ab.$$

Ex. 2. Multiply $-5ab^3x$ by $-ab^3x$.

Here the absolute value of the product is $5a^2b^6x^2$, and by the rule of signs the product is positive.

$$\therefore (-5ab^3x)\times(-ab^3x)=5a^2b^6x^2.$$

Ex. 3. Find the continued product of $3a^2b$, $-2a^3b^2$, $-ab^4$.

$3a^2b\times(-2a^3b^2)=-6a^5b^3$;

$(-6a^5b^3)\times(-ab^4)=+6a^6b^7$.

Thus the complete product is $6a^6b^7$.

This result, however, may be written down at once; for

$$3a^2b\times 2a^3b^2\times ab^4=6a^6b^7,$$

and by the rule of signs the required product is positive.

Ex. 4. Multiply $6a^3-5a^2b-4ab^2$ by $-3ab^2$.

The product is the algebraic sum of the partial products formed according to the rule enunciated in Art. 40;

thus $(6a^3-5a^2b-4ab^2)\times(-3ab^2)=-18a^4b^2+15a^3b^3+12a^2b^4$.

EXAMPLES IV. c.

Multiply together

1. ax and $-3ax$.

2. $-2abx$ and $-7abx$.

3. a^2b and $-ab^2$.

4. $6x^2y$ and $-10xy$.

5. $-abcd$ and $-3a^2b^3c^4d^5$.

6. xyz and $-5x^2y^3z$.

7. $3xy+4yz$ and $-12xyz$.

8. $ab-bc$ and a^2bc^3.

9. $-x-y-z$ and $-3x$.
10. $a^2-b^2+c^2$ and abc.
11. $-ab+bc-ca$ and $-abc$.
12. $-2a^2b-4ab^2$ and $-7a^2b^2$.
13. $5x^2y-6xy^2+8x^2y^2$ and $3xy$.
14. $-7x^3y-5xy^3$ and $-8x^3y^3$.
15. $-5xy^2z+3xyz^2-8x^2yz$ and xyz.
16. $4x^2y^2z^2-8xyz$ and $-12x^3yz^3$.
17. $-13xy^2-15x^2y$ and $-7x^3y^3$.
18. $8xyz-10x^3yz^3$ and $-xyz$.
19. $abc-a^2bc-ab^2c$ and $-abc$.
20. $-a^2bc+b^2ca-c^2ab$ and $-ab$.

Find the product of

21. $2a-3b+4c$ and $-\frac{3}{2}a$.
22. $3x-2y-4$ and $-\frac{5}{6}x$.
23. $\frac{2}{3}a-\frac{1}{6}b-c$ and $\frac{3}{8}ax$.
24. $\frac{6}{7}a^2x^2-\frac{3}{2}ax^3$ and $-\frac{7}{3}a^3x$.
25. $-\frac{5}{3}a^2x^2$ and $-\frac{3}{2}a^2+ax-\frac{3}{5}x^2$.
26. $-\frac{7}{2}xy$ and $-3x^2+\frac{2}{7}xy$.
27. $-\frac{3}{2}x^3y^2$ and $-\frac{1}{3}x^2+2y^2$.

47. The results of Art. 41 may be extended to the case where one or both of the expressions to be multiplied together contain more than two terms. For instance

$$(a-b+c)m=am-bm+cm;$$

replacing m by $x-y$, we have

$$\begin{aligned}(a-b+c)(x-y)&=a(x-y)-b(x-y)+c(x-y)\\&=(ax-ay)-(bx-by)+(cx-cy)\\&=ax-ay-bx+by+cx-cy.\end{aligned}$$

48. These results enable us to state the general rule for multiplying together any two compound expressions.

Rule. *Multiply each term of the first expression by each term of the second. When the terms multiplied together have like signs, prefix to the product the sign* $+$, *when unlike prefix* $-$; *the algebraic sum of the partial products so formed gives the complete product.*

This process is called **Distributing the Product.**

Ex. 1. Multiply $x+8$ by $x+7$.

$$\begin{array}{lr} & x+8 \\ & x+7 \\ \hline & x^2+8x \qquad \\ & +7x+56 \\ \hline \text{by addition} & x^2+15x+56 \end{array}$$

NOTE. We begin on the left and work to the right, placing the second result one place to the right, so that like terms may stand in the same vertical column.

Ex. 2. Multiply $2x - 3y$ by $4x - 7y$.

$$\begin{array}{lll} 2x & -3y & \\ 4x & -7y & \\ \hline 8x^2 & -12xy & \\ & -14xy & +21y^2 \\ \hline 8x^2 & -26xy & +21y^2 \end{array}$$

by addition $8x^2 - 26xy + 21y^2$

EXAMPLES IV. d.

Find the product of

1. $x + 5$ and $x + 10$.
2. $x + 5$ and $x - 5$.
3. $x - 7$ and $x - 10$.
4. $x - 7$ and $x + 10$.
5. $x + 7$ and $x - 10$.
6. $x + 7$ and $x + 10$.
7. $x + 6$ and $x - 6$.
8. $x + 8$ and $x - 4$.
9. $x - 12$ and $x - 1$.
10. $x + 12$ and $x - 1$.
11. $x - 15$ and $x + 15$.
12. $x - 15$ and $-x + 3$.
13. $-x - 2$ and $-x - 3$.
14. $-x + 7$ and $x - 7$.
15. $-x + 5$ and $-x - 5$.
16. $x - 13$ and $x + 14$.
17. $x - 17$ and $x + 18$.
18. $x + 19$ and $x - 20$.
19. $-x - 16$ and $-x + 16$.
20. $-x + 21$ and $x - 21$.
21. $2x - 3$ and $x + 8$.
22. $2x + 3$ and $x - 8$.
23. $x - 5$ and $2x - 1$.
24. $2x - 5$ and $x - 1$.
25. $3x - 5$ and $2x + 7$.
26. $3x + 5$ and $2x - 7$.
27. $5x - 6$ and $2x + 3$.
28. $5x + 6$ and $2x - 3$.
29. $3x - 5y$ and $3x + 5y$.
30. $3x - 5y$ and $3x - 5y$.
31. $a - 2b$ and $a + 3b$.
32. $a - 7b$ and $a + 8b$.
33. $3a - 6b$ and $a - 8b$.
34. $a - 9b$ and $a + 5b$.
35. $x + a$ and $x - b$.
36. $x - a$ and $x + b$.
37. $x - 2a$ and $x + 3b$.
38. $ax - by$ and $ax + by$.
39. $xy - ab$ and $xy + ab$.
40. $2pq - 3r$ and $2pq + 3r$.

49. We shall now give a few examples of greater difficulty.

Ex. **1**. Find the product of $3x^2 - 2x - 5$ and $2x - 5$.

$$\begin{array}{rrrr}
3x^2 & -\ 2x & -\ 5 & \\
2x & -\ 5 & & \\
\hline
6x^3 - & 4x^2 & -\ 10x & \\
 & -\ 15x^2 & +\ 10x & +\ 25 \\
\hline
6x^3 - & 19x^2 & & +\ 25
\end{array}$$

Each term of the first expression is multiplied by $2x$, the first term of the second expression; then each term of the first expression is multiplied by -5; like terms are placed in the same columns and the results added.

Ex. **2**. Multiply $a - b + 3c$ by $a + 2b$

$$\begin{array}{lllll}
a^2 & -\ b & +\ 3c & & \\
a & +\ 2b & & & \\
\hline
a^2 & -\ ab & +\ 3ac & & \\
 & \ 2ab & & -\ 2b^2 & +\ 6bc \\
\hline
a^2 & +\ ab & +\ 3ac & -\ 2b^2 & +\ 6bc
\end{array}$$

When the coefficients are fractional, we use the ordinary process of Multiplication, combining the fractional coefficients by the rules of Arithmetic.

Ex. **3**. Multiply $\frac{1}{3}a^2 - \frac{1}{2}ab + \frac{2}{3}b^2$ by $\frac{1}{2}a + \frac{1}{3}b$.

$$\begin{array}{llll}
\frac{1}{3}a^2 & -\ \frac{1}{2}ab & +\ \frac{2}{3}b^2 & \\
\frac{1}{2}a & +\ \frac{1}{3}b & & \\
\hline
\frac{1}{6}a^3 & -\ \frac{1}{4}a^2b & +\ \frac{1}{3}ab^2 & \\
 & +\ \frac{1}{9}a^2b & -\ \frac{1}{6}ab^2 & +\ \frac{2}{9}b^3 \\
\hline
\frac{1}{6}a^3 & -\ \frac{5}{36}a^2b & +\ \frac{1}{6}ab^2 & +\ \frac{2}{9}b^3
\end{array}$$

50. If the expressions are not arranged according to powers ascending or descending of some common letter, a rearrangement will be found convenient.

Ex. Multiply $2xz - z^2 + 2x^2 - 3yz + xy$ by $x - y + 2z$.

$$\begin{array}{lllllllll}
2x^2 & +\ xy & +\ 2xz & -\ 3yz & -\ z^2 & & & & \\
x & -\ y & +\ 2z & & & & & & \\
\hline
2x^3 & +\ x^2y & +\ 2x^2z & -\ 3xyz & -\ xz^2 & & & & \\
 & -\ 2x^2y & & -\ 2xyz & & -\ xy^2 & +\ 3y^2z & +\ yz^2 & \\
 & & 4x^2z & +\ 2xyz & +\ 4xz^2 & & & -\ 6yz^2 & -\ 2z^3 \\
\hline
2x^3 & -\ x^2y & +\ 6x^2z & -\ 3xyz & +\ 3xz^2 & -\ xy^2 & +\ 3y^2z & -\ 5yz^2 & -\ 2z^3
\end{array}$$

EXAMPLES IV. e.

Multiply together

1. $a+b+c$ and $a+b-c$.
2. $a-2b+c$ and $a+2b-c$.
3. a^2-ab+b^2 and a^2+ab+b^2.
4. x^2+3y^2 and $x+4y$.
5. x^3-2x^2+8 and $x+2$.
6. $x^4-x^2y^2+y^4$ and x^2+y^2.
7. x^2+xy+y^2 and $x-y$.
8. $a^2-2ax+4x^2$ and $a^2+2ax+4x^2$.
9. $16a^2+12ab+9b^2$ and $4a-3b$.
10. $a^2x-ax^2+x^3-a^3$ and $x+a$.
11. x^2+x-2 and x^2+x-6.
12. $2x^3-3x^2+2x$ and $2x^2+3x+2$.
13. $-a^5+a^4b-a^3b^2$ and $-a-b$.
14. x^3-7x+5 and x^2-2x+3.
15. $a^3+2a^2b+2ab^2$ and $a^2-2ab+2b^2$.
16. $4x^2+6xy+9y^2$ and $2x-3y$.
17. $x^2-3xy-y^2$ and $-x^2+xy+y^2$.
18. $b^3-a^2b^2+a^3$ and $a^3+a^2b^2+b^3$.
19. $x^2-2xy+y^2$ and $x^2+2xy+y^2$.
20. $ab+cd+ac+bd$ and $ab+cd-ac-bd$.
21. $-3a^2b^2+4ab^3+15a^3b$ and $5a^2b^2+ab^3-3b^4$.
22. $27x^3-36ax^2+48a^2x-64a^3$ and $3x+4a$.
23. $a^2-5ab-b^2$ and $a^2+5ab+b^2$.
24. $x^2-xy+x+y^2+y+1$ and $x+y-1$.
25. $a^2+b^2+c^2-bc-ca-ab$ and $a+b+c$.
26. $-x^3y+y^4+x^2y^2+x^4-xy^3$ and $x+y$.
27. $x^{12}-x^9y^2+x^6y^4-x^3y^6+y^8$ and x^3+y^2.
28. $3a^2+2a+2a^3+1+a^4$ and a^2-2a+1.
29. $-ax^2+3axy^2-9ay^4$ and $-ax-3ay^2$.
30. $-2x^3y+y^4+3x^2y^2+x^4-2xy^3$ and $x^2+2xy+y^2$.
31. $\frac{1}{2}a^2+\frac{1}{3}a+\frac{1}{4}$ and $\frac{1}{2}a-\frac{1}{3}$.
32. $\frac{1}{2}x^2-2x+\frac{3}{2}$ and $\frac{1}{2}x+\frac{1}{3}$.

33. $\frac{2}{3}x^2 + xy + \frac{3}{2}y^2$ and $\frac{1}{3}x - \frac{1}{2}y$.

34. $\frac{3}{2}x^2 - ax - \frac{2}{3}a^2$ and $\frac{3}{4}x^2 - \frac{1}{2}ax + \frac{1}{3}a^2$.

35. $\frac{1}{2}x^2 - \frac{2}{3}x - \frac{3}{4}$ and $\frac{1}{2}x^2 + \frac{2}{3}x - \frac{3}{4}$.

36. $\frac{2}{3}ax + \frac{2}{3}x^2 + \frac{1}{3}a^2$ and $\frac{3}{4}a^2 + \frac{3}{2}x^2 - \frac{3}{2}ax$.

NOTE. Examples involving literal, fractional, and negative exponents will be found in the chapter on the Theory of Indices.

51. Products Written by Inspection. Although the result of multiplying together two binomial factors, such as $x + 8$ and $x - 7$, can always be obtained by the methods already explained, it is of the utmost importance that the student should learn to write the product rapidly *by inspection.*

This is done by observing in what way the coefficients of the terms in the product arise, and noticing that they result from the combination of the numerical coefficients in the two binomials which are multiplied together; thus

$$\begin{aligned}(x+8)(x+7) &= x^2 + 8x + 7x + 56\\ &= x^2 + 15x + 56.\\ (x-8)(x-7) &= x^2 - 8x - 7x + 56\\ &= x^2 - 15x + 56.\\ (x+8)(x-7) &= x^2 + 8x - 7x - 56\\ &= x^2 + x - 56.\\ (x-8)(x+7) &= x^2 - 8x + 7x - 56\\ &= x^2 - x - 56.\end{aligned}$$

In each of these results we notice that:

1. The product consists of three terms.
2. The first term is the product of the first terms of the two binomial expressions.
3. The third term is the product of the second terms of the two binomial expressions.
4. The middle term has for its coefficient the sum of the numerical quantities (taken with their proper signs) in the second terms of the two binomial expressions.

The intermediate step in the work may be omitted, and the products written at once, as in the following examples:

$$(x+2)(x+3)=x^2+5x+6.$$
$$(x-3)(x+4)=x^2+x-12.$$
$$(x+6)(x-9)=x^2-3x-54.$$
$$(x-4y)(x-10y)=x^2-14xy+40y^2.$$
$$(x-6y)(x+4y)=x^2-2xy-24y^2.$$

By an easy extension of these principles we may write the product of *any* two binomials.

Thus
$$\begin{aligned}(2x+3y)(x-y)&=2x^2+3xy-2xy-3y^2\\&=2x^2+xy-3y^2.\\(3x-4y)(2x+y)&=6x^2-8xy+3xy-4y^2\\&=6x^2-5xy-4y^2.\end{aligned}$$

EXAMPLES IV. f.

Write the values of the following products:

1. $(x+8)(x-5)$.
2. $(x+6)(x-1)$.
3. $(x-3)(x+10)$.
4. $(x-1)(x+5)$.
5. $(x+7)(x-9)$.
6. $(x-10)(x-8)$.
7. $(x-4)(x+11)$.
8. $(x-2)(x+4)$.
9. $(x+2)(x-2)$.
10. $(a-1)(a+1)$.
11. $(a+9)(a-5)$.
12. $(a-3)(a+12)$.
13. $(a-8)(a+4)$.
14. $(a-8)(a+8)$.
15. $(a-6)(a+13)$.
16. $(a+3)(a+3)$.
17. $(a-11)(a+11)$.
18. $(a-8)(a-8)$.
19. $(x-3a)(x+2a)$.
20. $(x+6a)(x-5a)$.
21. $(x+3a)(x-3a)$.
22. $(x+4y)(x-2y)$.
23. $(x+7y)(x-7y)$.
24. $(x-3y)(x-3y)$.
25. $(3x-1)(x+1)$.
26. $(2x+5)(2x-1)$.
27. $(3x+7)(2x-3)$.
28. $(4x-3)(2x+3)$.

29. $(3x+8)(3x-8)$.
30. $(2x-5)(2x-5)$.
31. $(3x-2y)(3x+y)$.
32. $(3x+2y)(3x+2y)$.
33. $(2x+7y)(2x-5y)$.
34. $(5x+3a)(5x-3a)$.
35. $(2x-5a)(x+5a)$.
36. $(2x+a)(2x+a)$.

MULTIPLICATION BY DETACHED COEFFICIENTS.

52. In the following cases we lessen the labor of multiplication by using the **Method of Detached Coefficients**:

(i.) When two compound expressions contain but one letter.

(ii.) When two compound expressions are homogeneous and contain but two letters.

Ex. **1.** Multiply $2x^3-4x^2+5x-5$ by $3x^2+4x-2$.

Writing coefficients only,

$$\begin{array}{l}
2-\ \ 4+\ \ 5-\ \ 5\\
\underline{3+\ \ 4-\ \ 2\qquad\quad}\\
6-12+15-15\\
\quad+\ \ 8-16+20-20\\
\underline{\qquad\quad-\ \ 4+\ \ 8-10+10}\\
6-\ \ 4-\ \ 5+13-30+10
\end{array}$$

Inserting the literal factors according to the law of their formation, which is readily seen, we have for the complete product,

$$6x^5-4x^4-5x^3+13x^2-30x+10.$$

Ex. **2.** Multiply $3a^4+2a^3b+4ab^3+2b^4$ by $2a^2-b^2$.

$$\begin{array}{l}
3+2+0+4+2\\
\underline{2+0-1\qquad\quad}\\
6+4+0+8+4\\
\underline{\qquad\quad-3-2-0-4-2}\\
6+4-3+6+4-4-2
\end{array}$$

In the first expression the term containing a^2b^2 is missing, so we write a zero in the corresponding term in the line of coefficients. In the second expression we write a zero for the coefficient of the missing term ab.

The law of formation of literal factors is readily seen, and we have for the complete product,

$$6a^6+4a^5b-3a^4b^2+6a^3b^3+4a^2b^4-4ab^5-2b^6.$$

EXAMPLES IV. g.

1. Multiply $x^5+x^4+x^2+2x+1$ by x^3+x-2.
2. Multiply $a^3+6a^2b+12ab^2+8b^3$ by $3a^3+2b^3$.
3. Multiply $2a^4-3a^2+4a+4$ by $2a^2-3a-2$.
4. Multiply $3x^5+2x^4y-x^3y^2+xy^4$ by $x^2+4xy-5y^2$.

CHAPTER V.

DIVISION.

53. When a quantity a is divided by the quantity b, the **quotient** is defined to be that which when multiplied by b produces a. The operation is denoted by $a \div b$, $\frac{a}{b}$, or a/b; in each of these modes of expression a is called the **dividend,** and b the **divisor.**

Division is thus the inverse of multiplication, and

$$(a \div b) \times b = a.$$

54. *The* **Rule of Signs** *holds for division.*

Thus

$$ab \div a = \frac{ab}{a} = \frac{a \times b}{a} = b.$$

$$-ab \div a = \frac{-ab}{a} = \frac{a \times (-b)}{a} = -b.$$

$$ab \div (-a) = \frac{ab}{-a} = \frac{(-a) \times (-b)}{-a} = -b.$$

$$-ab \div (-a) = \frac{-ab}{-a} = \frac{(-a) \times b}{-a} = b.$$

Hence in division as well as multiplication

like signs produce $+$,

unlike signs produce $-$.

55. Since Division is the inverse of Multiplication, it follows that the Laws of Commutation, Association, and Distribution, which have been established for Multiplication, hold for Division.

DIVISION OF SIMPLE EXPRESSIONS.

56. The method is shown in the following examples:

Ex. **1.** Since the product of 4 and x is $4x$, it follows that when $4x$ is divided by x the quotient is 4,
or otherwise, $$4x \div x = 4.$$

Ex. **2.** Divide $27a^5$ by $9a^3$.

The quotient $= \dfrac{27a^5}{9a^3} = \dfrac{27\,aaaaa}{9\,aaa}$
$= 3\,aa = 3\,a^2.$

We remove from the divisor and dividend the factors common to both, as in Arithmetic.

Therefore $$27a^5 \div 9a^3 = 3a^2.$$

Ex. **3.** Divide $35a^3b^2c^3$ by $7ab^2c^3$.

The quotient $$= \frac{35\,aaa\,.\,bb\,.\,ccc}{7\,a\,.\,bb\,.\,cc} = 5\,aa\,.\,c = 5a^2c.$$

We see, in each case, that *the index of any letter in the quotient is the difference of the indices of that letter in the dividend and divisor.* This is called the **Index Law for Division.**

We can now state the complete rule:

Rule. *The index of each letter in the quotient is obtained by subtracting the index of that letter in the divisor from that in the dividend.*

To the result so obtained prefix with its proper sign the quotient of the coefficient of the dividend by that of the divisor.

Ex. **4.** Divide $45a^6b^2x^4$ by $-9a^3bx^2$.

$$\text{The quotient } = (-5) \times a^{6-3}b^{2-1}x^{4-2}$$
$$= -5a^3bx^2.$$

Ex. **5.** $-21a^2b^3 \div (-7a^2b^2) = 3b.$

Note. If we apply the rule to divide any power of a letter by the same power of the letter, we are led to a curious conclusion.

Thus, by the rule $$a^3 \div a^3 = a^{3-3} = a^0;$$

but also $$a^3 \div a^3 = \frac{a^3}{a^3} = 1.$$
$$\therefore\ a^0 = 1.$$

This result will appear somewhat strange to the beginner, but its full significance will be explained in the chapter on the Theory of Indices.

DIVISION OF A COMPOUND EXPRESSION BY A SIMPLE EXPRESSION.

57. Rule. *To divide a compound expression by a single factor, divide each term separately by that factor, and take the algebraic sum of the partial quotients so obtained.*

This follows at once from Art. 40.

Ex. 1. $(9x - 12y + 3z) \div -3 = -3x + 4y - z$.

Ex. 2. $(36a^3b^2 - 24a^2b^5 - 20a^4b^2) \div 4a^2b = 9ab - 6b^4 - 5a^2b$.

Ex. 3. $(2x^2 - 5xy + \frac{3}{2}x^2y^3) \div -\frac{1}{2}x = -4x + 10y - 3xy^3$.

EXAMPLES V. a.

Divide

1. $3x^3$ by x^2.
2. $27x^4$ by $-9x^3$.
3. $-35x^6$ by $7x^3$.
4. x^3y^3 by x^2y.
5. a^4x^3 by $-a^2x^3$.
6. $12a^6b^6c^6$ by $-3a^4b^2c$.
7. $-a^5c^9$ by $-ac^3$.
8. $15x^5y^7z^4$ by $5x^2y^2z^2$.
9. $-16x^3y^2$ by $-4xy^2$.
10. $-48a^9$ by $-8a^3$.
11. $63a^7b^8c^3$ by $9a^5b^5c^3$.
12. $7a^2bc$ by $-7a^2bc$.
13. $28a^4b^3$ by $-4a^3b$.
14. $16b^2yx^2$ by $-2xy$.
15. $-50y^3x^3$ by $-5x^3y$.
16. $x^3 - 3x^2 + x$ by x.
17. $x^6 - 7x^5 + 4x^4$ by x^2.
18. $10x^7 - 8x^6 + 3x^4$ by x^3.
19. $15x^5 - 25x^4$ by $-5x^3$.
20. $-24x^6 - 32x^4$ by $-8x^3$.
21. $34x^3y^2 - 51x^2y^3$ by $17xy$.
22. $a^2 - ab - ac$ by $-a$.
23. $a^3 - a^2b - a^2b^2$ by a^2.
24. $3x^3 - 9x^2y - 12xy^2$ by $-3x$.
25. $4x^4y^4 - 8x^3y^2 + 6xy^3$ by $-2xy$.
26. $\frac{1}{2}x^5y^2 - 3x^3y^4$ by $-\frac{3}{2}x^3y^2$.
27. $-\frac{5}{2}x^2 + \frac{5}{3}xy + \frac{10}{3}x$ by $-\frac{5}{6}x$.
28. $-2a^5x^3 + \frac{7}{2}a^4x^4$ by $\frac{7}{3}a^3x$.

DIVISION OF COMPOUND EXPRESSIONS.

58. We employ the following rule:

Rule. 1. *Arrange divisor and dividend according to ascending or descending powers of some common letter.*

2. *Divide the term on the left of the dividend by the term on the left of the divisor, and put the result in the quotient.*

3. *Multiply the* WHOLE *divisor by this quotient, and put the product under the dividend.*

4. *Subtract and bring down from the dividend as many terms as may be necessary.*

Repeat these operations till all the terms from the dividend are brought down.

Ex. **1.** Divide $x^2 + 11x + 30$ by $x + 6$.

Arrange the work thus:

$$x + 6)x^2 + 11x + 30($$

divide x^2, the first term of the dividend, by x, the first term of the divisor; the quotient is x. Multiply the *whole* divisor by x, and put the product $x^2 + 6x$ under the dividend. We then have

$$\begin{array}{rl} x + 6)x^2 + 11x + 30(x \\ \underline{x^2 + \ \ 6x} \qquad \\ \text{by subtraction} \qquad 5x + 30 \end{array}$$

On repeating the process above explained we find that the next term in the quotient is $+5$.

The entire operation is more compactly written as follows:

$$\begin{array}{r} x + 6)x^2 + 11x + 30(x + 5 \\ \underline{x^2 + \ \ 6x} \qquad\qquad \\ 5x + 30 \qquad \\ \underline{5x + 30} \qquad \end{array}$$

The reason for the rule is this: the dividend may be divided into as many parts as may be convenient, and the complete quotient is found by taking the sum of all the partial quotients. Thus $x^2 + 11x + 30$ is divided by the above process into two parts, namely, $x^2 + 6x$, and $5x + 30$, and each of these is divided by $x + 6$; thus we obtain the complete quotient $x + 5$.

Ex. **2.** Divide $24x^2 - 65xy + 21y^2$ by $8x - 3y$.

$$\begin{array}{r} 8x - 3y)24x^2 - 65xy + 21y^2(3x - 7y. \\ \underline{24x^2 - \ \ 9xy} \qquad\qquad\qquad \\ -56xy + 21y^2 \qquad\qquad \\ \underline{-56xy + 21y^2} \qquad\qquad \end{array}$$

EXAMPLES V. b.

Divide

1. x^2+3x+2 by $x+1$.
2. $x^2-7x+12$ by $x-3$.
3. $a^2-11a+30$ by $a-5$.
4. $a^2-49a+600$ by $a-25$.
5. $3x^2+10x+3$ by $x+3$.
6. $2x^2+11x+5$ by $2x+1$.
7. $5x^2+11x+2$ by $x+2$.
8. $2x^2+17x+21$ by $2x+3$.
9. $4x^2+23x+15$ by $4x+3$.
10. $6x^2-7x-3$ by $2x-3$.
11. $3x^2+x-14$ by $x-2$.
12. $3x^2-x-14$ by $x+2$.
13. $6x^2-31x+35$ by $2x-7$.
14. $12a^2-7ax-12x^2$ by $3a-4x$.
15. $15a^2+17ax-4x^2$ by $3a+4x$.
16. $12a^2-11ac-36c^2$ by $4a-9c$.
17. $-4xy-15y^2+96x^2$ by $12x-5y$.
18. $7x^3+96x^2-28x$ by $7x-2$.
19. $100x^3-3x-13x^2$ by $3+25x$.
20. $27x^3+9x^2-3x-10$ by $3x-2$.
21. $16a^3-46a^2+39a-9$ by $8a-3$.

59. The process of Art. 58 is applicable to cases in which the divisor consists of more than two terms.

Ex. 1. Divide $6x^5-x^4+4x^3-5x^2-x-15$ by $2x^2-x+3$.

$$
\begin{array}{l}
2x^2-x+3)\,6x^5-x^4+4x^3-5x^2-x-15\,(3x^3+x^2-2x-5 \\
\qquad\qquad\underline{6x^5-3x^4+9x^3} \\
\qquad\qquad\qquad\quad 2x^4-5x^3-5x^2 \\
\qquad\qquad\qquad\quad \underline{2x^4-x^3+3x^2} \\
\qquad\qquad\qquad\qquad -4x^3-8x^2-x \\
\qquad\qquad\qquad\qquad \underline{-4x^3+2x^2-6x} \\
\qquad\qquad\qquad\qquad\qquad -10x^2+5x-15 \\
\qquad\qquad\qquad\qquad\qquad \underline{-10x^2+5x-15}
\end{array}
$$

Ex. 2. Divide $a^3+b^3+c^3-3abc$ by $a+b+c$.

$$
\begin{array}{l}
a+b+c)\,a^3-3abc+b^3+c^3\,(a^2-ab-ac+b^2-bc+c^2 \\
\qquad\qquad\underline{a^3+a^2b+a^2c} \\
\qquad\qquad -a^2b-a^2c-3abc \\
\qquad\qquad \underline{-a^2b-ab^2-abc} \\
\qquad\qquad\qquad -a^2c+ab^2-2abc \\
\qquad\qquad\qquad \underline{-a^2c-abc-ac^2} \\
\qquad\qquad\qquad\qquad ab^2-abc+ac^2+b^3 \\
\qquad\qquad\qquad\qquad \underline{ab^2+b^3+b^2c} \\
\qquad\qquad\qquad\qquad\quad -abc+ac^2-b^2c \\
\qquad\qquad\qquad\qquad\quad \underline{-abc-b^2c-bc^2} \\
\qquad\qquad\qquad\qquad\qquad ac^2+bc^2+c^3 \\
\qquad\qquad\qquad\qquad\qquad \underline{ac^2+bc^2+c^3}
\end{array}
$$

NOTE. The result of this division will be referred to later.

60. Sometimes it will be found convenient to arrange the expressions in *ascending* powers of some common letter.

Ex. Divide $2a^3 + 10 - 16a - 39a^2 + 15a^4$ by $2 - 4a - 5a^2$.

$$\begin{array}{r|l} 2 - 4a - 5a^2) & 10 - 16a - 39a^2 + 2a^3 + 15a^4 \,(5 + 2a - 3a^2 \\ & \underline{10 - 20a - 25a^2} \\ & \quad 4a - 14a^2 + 2a^3 \\ & \quad \underline{4a - 8a^2 - 10a^3} \\ & \quad\quad - 6a^2 + 12a^3 + 15a^4 \\ & \quad\quad \underline{- 6a^2 + 12a^3 + 15a^4} \end{array}$$

61. When the coefficients are fractional, the ordinary process may still be employed.

Ex. Divide $\frac{1}{4}x^3 + \frac{1}{72}xy^2 + \frac{1}{12}y^3$ by $\frac{1}{2}x + \frac{1}{3}y$.

$$\begin{array}{r|l} \frac{1}{2}x + \frac{1}{3}y) & \frac{1}{4}x^3 + \frac{1}{72}xy^2 + \frac{1}{12}y^3 \,(\frac{1}{2}x^2 - \frac{1}{3}xy + \frac{1}{4}y^2 \\ & \underline{\frac{1}{4}x^3 + \frac{1}{6}x^2y} \\ & \quad - \frac{1}{6}x^2y + \frac{1}{72}xy^2 \\ & \quad \underline{- \frac{1}{6}x^2y - \frac{1}{9}xy^2} \\ & \quad\quad \frac{1}{8}xy^2 + \frac{1}{12}y^3 \\ & \quad\quad \underline{\frac{1}{8}xy^2 + \frac{1}{12}y^3} \end{array}$$

In the examples given hitherto the divisor has been exactly contained in the dividend. When the division is not exact, the work should be carried on until the remainder is of lower dimensions [Art. 29] than the divisor.

EXAMPLES V. c.

Divide

1. $x^3 - x^2 - 9x - 12$ by $x^2 + 3x + 3$.
2. $2y^3 - 3y^2 - 6y - 1$ by $2y^2 - 5y - 1$.
3. $6m^3 - m^2 - 14m + 3$ by $3m^2 + 4m - 1$.
4. $6a^5 - 13a^4 + 4a^3 + 3a^2$ by $3a^3 - 2a^2 - a$.
5. $x^4 + x^3 + 7x^2 - 6x + 8$ by $x^2 + 2x + 8$.
6. $a^4 - a^3 - 8a^2 + 12a - 9$ by $a^2 + 2a - 3$.
7. $a^4 + 6a^3 + 13a^2 + 12a + 4$ by $a^2 + 3a + 2$.
8. $2x^4 - x^3 + 4x^2 + 7x + 1$ by $x^2 - x + 3$.
9. $x^5 - 5x^4 + 9x^3 - 6x^2 - x + 2$ by $x^2 - 3x + 2$.
10. $x^5 - 4x^4 + 3x^3 + 3x^2 - 3x + 2$ by $x^2 - x - 2$.

11. $30x^4 + 11x^3 - 82x^2 - 5x + 3$ by $2x - 4 + 3x^2$.
12. $30y + 9 - 71y^3 + 28y^4 - 35y^2$ by $4y^2 - 13y + 6$.
13. $6k^5 - 15k^4 + 4k^3 + 7k^2 - 7k + 2$ by $3k^3 - k + 1$.
14. $15 + 2m^4 - 31m + 9m^2 + 4m^3 + m^5$ by $3 - 2m - m^2$.
15. $2x^3 - 8x + x^4 + 12 - 7x^2$ by $x^2 + 2 - 3x$.
16. $x^5 - 2x^4 - 4x^3 + 19x^2$ by $x^3 - 7x + 5$.
17. $192 - x^4 + 128x + 4x^2 - 8x^3$ by $16 - x^2$.
18. $14x^4 + 45x^3y + 78x^2y^2 + 45xy^3 + 14y^4$ by $2x^2 + 5xy + 7y^2$.
19. $x^5 - x^4y + x^3y^2 - x^3 + x^2 - y^3$ by $x^3 - x - y$.
20. $x^5 + x^4y - x^3y^2 + x^3 - 2xy^2 + y^3$ by $x^2 + xy - y^2$.
21. $a^9 - b^9$ by $a^3 - b^3$.
22. $x^9 - y^9$ by $x^2 + xy + y^2$.
23. $x^7 - 2y^{14} - 7x^5y^4 - 7xy^{12} + 14x^3y^8$ by $x - 2y^2$.
24. $a^3 + 3a^2b + b^3 - 1 + 3ab^2$ by $a + b - 1$.
25. $x^8 - y^8$ by $x^3 + x^2y + xy^2 + y^3$.
26. $a^{12} - b^{12}$ by $a^2 - b^2$.
27. $a^{12} + 2a^6b^6 + b^{12}$ by $a^4 + 2a^2b^2 + b^4$.
28. $1 - a^3 - 8x^3 - 6ax$ by $1 - a - 2x$.

Find the quotient of

29. $\frac{1}{8}a^3 - \frac{9}{4}a^2x + \frac{27}{2}ax^2 - 27x^3$ by $\frac{1}{2}a - 3x$.
30. $\frac{1}{27}a^3 - \frac{1}{12}a^2 + \frac{1}{16}a - \frac{1}{64}$ by $\frac{1}{3}a - \frac{1}{4}$.
31. $\frac{3}{4}a^2c^3 + \frac{6}{125}a^5$ by $\frac{1}{5}a^2 + \frac{1}{2}ac$.
32. $\frac{8}{27}a^5 - \frac{243}{512}ax^4$ by $\frac{2}{3}a - \frac{3}{4}x$.
33. $\frac{9}{16}a^4 - \frac{3}{4}a^3 - \frac{7}{4}a^2 + \frac{4}{3}a + \frac{16}{9}$ by $\frac{3}{2}a^2 - \frac{8}{3} - a$.
34. $36x^2 + \frac{1}{9}y^2 + \frac{1}{4} - 4xy - 6x + \frac{1}{3}y$ by $6x - \frac{1}{3}y - \frac{1}{2}$.

62. Important Cases in Division. The following examples in division may be easily verified; they are of great importance and should be carefully noticed.

$$\text{I.}\quad \begin{cases} \dfrac{x^2 - y^2}{x - y} = x + y, \\[2ex] \dfrac{x^3 - y^3}{x - y} = x^2 + xy + y^2, \\[2ex] \dfrac{x^4 - y^4}{x - y} = x^3 + x^2y + xy^2 + y^3, \end{cases}$$

and so on; the divisor being $x-y$, the terms in the quotient *all positive*, and the index in the dividend *either odd or even*.

$$\text{II.}\quad\begin{cases}\dfrac{x^3+y^3}{x+y}=x^2-xy+y^2,\\[2mm]\dfrac{x^5+y^5}{x+y}=x^4-x^3y+x^2y^2-xy^3+y^4,\\[2mm]\dfrac{x^7+y^7}{x+y}=x^6-x^5y+x^4y^2-x^3y^3+x^2y^4-xy^5+y^6,\end{cases}$$

and so on; the divisor being $x+y$, the terms in the quotient *alternately positive and negative*, and the index in the dividend *always odd.*

$$\text{III.}\quad\begin{cases}\dfrac{x^2-y^2}{x+y}=x-y,\\[2mm]\dfrac{x^4-y^4}{x+y}=x^3-x^2y+xy^2-y^3,\\[2mm]\dfrac{x^6-y^6}{x+y}=x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5,\end{cases}$$

and so on; the divisor being $x+y$, the terms in the quotient *alternately positive and negative*, and the index in the dividend *always even.*

IV. The expressions x^2+y^2, x^4+y^4, $x^6+y^6 \cdots$ (where the index is *even*, and the terms *both positive*), are *never* divisible by $x+y$ or $x-y$.

All these different cases may be more concisely stated as follows:

(1) x^n-y^n is divisible by $x-y$ if n be *any* whole number.

(2) x^n+y^n is divisible by $x+y$ if n be any *odd* whole number.

(3) x^n-y^n is divisible by $x+y$ if n be any *even* whole number.

(4) x^n+y^n is never divisible by $x+y$ or $x-y$ when n is an *even* whole number.

NOTE. General proofs of these statements will be found in Art. 106.

DIVISION BY DETACHED COEFFICIENTS.

63. In Art. 52 we considered certain cases of compound expressions in which the work of multiplication could be shortened by using the *Method of Detached Coefficients.* In the same cases the labor of division can be considerably abridged by using detached coefficients, and employing an arrangement of terms known as **Horner's Method of Synthetic Division.** The following examples illustrate the method:

Ex. **1.** Divide $3x^5-8x^4-5x^3+26x^2-28x+24$ by x^3-2x^2-4x+8.

$$\begin{array}{r|rrrrrrl}
1 & 3 & -8 & -5 & +26 & -28 & +24 & \text{Dividend}\\
\text{Divisor}\quad 2 & & 6 & +12 & -24 & & & \\
4 & & & -4 & -8 & +16 & & \\
-8 & & & & +6 & +12 & -24 & \\ \hline
\text{Quotient} & 3 & -2 & +3 & +0 & +0 & +0 &
\end{array}$$

Inserting the literal factors in the quotient according to the law of their formation, which is readily seen, we have for a complete quotient, $3x^2-2x+3$.

Explanation. The column of figures to the left of the vertical line consists of the coefficients of the divisor, *the sign of each after the first being changed*, which enables us *to replace the process of subtraction by that of addition* at each successive stage of the work. Dividing the first term of the dividend by the first term of the divisor, we obtain 3, the first term of the quotient. Multiplying 2, 4, and -8, the remaining terms of the divisor, by this first term of the quotient gives the *second horizontal line.* We then *add* the terms in the second column to the right of the vertical line and obtain -2, which is the coefficient of the second term of the quotient. With this coefficient as a multiplier, and using 2, 4, and -8 again as a multiplicand, we form the *third horizontal line.* Adding the terms in the third column gives 3, which is the third term of the quotient. With this coefficient as a multiplier and the same multiplicand as before, we form the *fourth horizontal line.* As only zeros now appear in the quotient, the division is exact.

Ex. **2.** Divide $2a^7+7a^6b+12a^5b^2+10a^4b^3-4a^3b^4$ by $2a^3+3a^2b-b^3$ to *four terms* in the quotient.

$$\begin{array}{r|rrrr|lll}
2 & 2 & +7 & +12 & +10 & -4a^3b^4 & +0a^2b^5 & +0ab^6\\
-3 & & -3 & +0 & +1 & & & \\
0 & & & -6 & +0 & +2 & & \\
1 & & & & -9 & +0 & +3 & \\
 & & & & & -3 & +0 & +1\\ \hline
\text{Quotient} & 1 & +2 & +3 & +1 & -5 & +3 & +1
\end{array}$$

Inserting literal factors, $a^4 + 2a^3b + 3a^2b^2 + ab^3$ is the complete quotient, and $-5a^3b^4 + 3a^2b^5 + ab^6$ is the remainder.

Explanation. The term ab^2 in the divisor is missing, so we write 0 for the coefficient of this term in the column of figures on the left of the vertical line. We add the columns as in Ex. 1, but as the first term of the divisor is 2, *we divide each sum by* 2 *before placing the result in the line of quotients.* We then use these quotients as multipliers, the multiplicand being in each case -3, 0, and 1, and form the horizontal lines as in Ex. 1. Having obtained the required number of terms in the *quotient*, the remainder is found by adding the rest of the columns and setting down the results *without dividing by* 2. By continuing the first horizontal line (dividend), as shown in this example, we at once see what literal factors the remainder must contain.

EXAMPLES V. d.

Divide:

1. $a^4 - 4a^3 + 2a^2 + 4a + 1$ by $a^2 - 2a - 1$.

2. $a^4 - 4a^3b + 6a^2b^2 + b^4 - 4ab^3$ by $a^2 + b^2 - 2ab$.

3. $a^5 - 10a^4b + 16a^3b^2 - 12a^2b^3 + ab^4 + 2b^5$ by $(a-b)^2$.

4. $x^8 - 2b^4x^4 + b^8$ by $x^3 + bx^2 + b^2x + b^3$.

5. $x^5 - 3x^2y^3 + 8xy^4 - 5y^5$ by $x^2 - 4xy + y^2$ to four terms in the quotient.

CHAPTER VI.

Removal and Insertion of Brackets.

64. We frequently find it necessary to enclose within brackets part of an expression already enclosed within brackets. For this purpose it is usual to employ brackets of different forms. The brackets in common use are ( ), { }, [ ]. Sometimes a line called a **vinculum** is drawn over the symbols to be connected; thus $a - \overline{b + c}$ is used with the same meaning as $a - (b + c)$, and hence

$$a - \overline{b + c} = a - b - c.$$

65. To **remove brackets** it is usually best to begin with the inside pair, and in dealing with each pair in succession we apply the rules already given in Arts. 25, 26.

Ex. **1.** Simplify, by removing brackets, the expression

$$a - 2b - [4a - 6b - \{3a - c + (5a - 2b - \overline{3a - c + 2b})\}].$$

Removing the brackets one by one, we have

$$\begin{aligned} & a - 2b - [4a - 6b - \{3a - c + (5a - 2b - 3a + c - 2b)\}] \\ &= a - 2b - [4a - 6b - \{3a - c + 5a - 2b - 3a + c - 2b\}] \\ &= a - 2b - [4a - 6b - 3a + c - 5a + 2b + 3a - c + 2b] \\ &= a - 2b - 4a + 6b + 3a - c + 5a - 2b - 3a + c - 2b \\ &= 2a, \text{ by collecting like terms.} \end{aligned}$$

Ex. **2.** Simplify the expression

$$-[-2x - \{3y - (2x - 3y) + (3x - 2y)\} + 2x].$$

$$\begin{aligned} \text{The expression } &= -[-2x - \{3y - 2x + 3y + 3x - 2y\} + 2x] \\ &= -[-2x - 3y + 2x - 3y - 3x + 2y + 2x] \\ &= 2x + 3y - 2x + 3y + 3x - 2y - 2x \\ &= x + 4y. \end{aligned}$$

EXAMPLES VI. a.

Simplify by removing brackets:

1. $a-(b-c)+a+(b-c)+b-(c+a)$.
2. $a-[b+\{a-(b+a)\}]$.
3. $a-[2a-\{3b-(4c-2a)\}]$.
4. $\{a-(b-c)\}+\{b-(c-a)\}-\{c-(a-b)\}$.
5. $2a-(5b+[3c-a])-(5a-[b+c])$.
6. $-\{-[-(a-\overline{b-c})]\}$.
7. $-(-(-(-x)))-(-(-y))$.
8. $-[a-\{b-(c-a)\}]-[b-\{c-(a-b)\}]$.
9. $-[-\{-(b+c-a)\}]+[-\{-(c+a-b)\}]$.
10. $-5x-[3y-\{2x-(2y-x)\}]$.
11. $-(-(-a))-(-(-(-x)))$.
12. $3a-[a+b-\{a+b+c-(a+b+c+d)\}]$.
13. $-2a-[3x+\{3c-(4y+3x+2a)\}]$.
14. $3x-[5y-\{6z-(4x-7y)\}]$.
15. $-[5x-(11y-3x)]-[5y-(3x-6y)]$.
16. $-[15x-\{14y-(15z+12y)-(10x-15z)\}]$.
17. $8x-\{16y-[3x-(12y-x)-8y]+x\}$.
18. $-[x-\{z+(x-z)-(z-x)-z\}-x]$.
19. $-[a+\{a-(a-x)-(a+x)-a\}-a]$.
20. $-[a-\{a+(x-a)-(x-a)-a\}-2a]$.

66. A coefficient placed before any bracket indicates that every term of the expression within the bracket is to be multiplied by that coefficient.

NOTE. The line between the numerator and denominator of a fraction is a kind of vinculum. Thus $\frac{x-5}{3}$ is equivalent to $\frac{1}{3}(x-5)$.

Again, an expression of the form $\sqrt{(x+y)}$ is often written $\sqrt{x+y}$, the line above being regarded as a vinculum indicating the square root of the compound expression $x+y$ *taken as a whole.*

Thus $$\sqrt{25+144}=\sqrt{169}=13,$$
whereas $$\sqrt{25}+\sqrt{144}=5+12=17.$$

67. Sometimes it is advisable to simplify in the course of the work.

Ex. Find the value of

$$84 - 7[-11x - 4\{-17x + 3(8 - \overline{9 - 5x})\}].$$

$$\begin{aligned}
\text{The expression} &= 84 - 7[-11x - 4\{-17x + 3(8 - 9 + 5x)\}] \\
&= 84 - 7[-11x - 4\{-17x + 3(5x - 1)\}] \\
&= 84 - 7[-11x - 4\{-17x + 15x - 3\}] \\
&= 84 - 7[-11x - 4\{-2x - 3\}] \\
&= 84 - 7[-11x + 8x + 12] \\
&= 84 - 7[-3x + 12] \\
&= 84 + 21x - 84 \\
&= 21x.
\end{aligned}$$

When the beginner has had a little practice, the number of steps may be considerably diminished.

EXAMPLES VI. b.

Simplify by removing brackets:

1. $a - [2b + \{3c - 3a - (a + b)\} + 2a - (b + 3c)]$.
2. $a + b - (c + a - [b + c - (a + b - \{c + a - (b + c - a)\})])$.
3. $a - (b - c) - [a - b - c - 2\{b + c - 3(c - a) - d\}]$.
4. $2x - (3y - 4z) - \{2x - (3y + 4z)\} - \{3y - (4z + 2x)\}$.
5. $b + c - (a + b - [c + a - (b + c - \{a + b - (c + a - b)\})])$.
6. $a - (b - c) - [a - b - c - 2\{b + c\}]$.
7. $3a^2 - [6a^2 - \{8b^2 - (9c^2 - 2a^2)\}]$.
8. $b - (c - a) - [b - a - c - 2\{c + a - 3(a - b) - d\}]$.
9. $-20(a - d) + 3(b - c) - 2[b + c + d - 3\{c + d - 4(d - a)\}]$.
10. $-4(a + d) + 24(b - c) - 2[c + d + a - 3\{d + a - 4(b + c)\}]$.
11. $-10(a + b) - [c + a + b - 3\{a + 2b - (c + a - b)\}] + 4c$.
12. $a - 2(b - c) - [-\{-(4a - b - c - 2\{a + b + c\})\}]$.
13. $2(3b - 5a) - 7[a - 6\{2 - 5(a - b)\}]$.
14. $6\{a - 2[b - 3(c + d)]\} - 4\{a - 3[b - 4(c - d)]\}$.
15. $5\{a - 2[a - 2(a + x)]\} - 4\{a - 2[a - 2(a + x)]\}$.
16. $-10\{a - 6[a - (b - c)]\} + 60\{b - (c + a)\}$.
17. $-3\{-2[-4(-a)]\} + 5\{-2[-2(-a)]\}$.
18. $-2\{-[-(x - y)]\} + \{-2[-(x - y)]\}$.

19. $\frac{1}{4}\{a-5(b-a)\}-\frac{3}{2}\left\{\frac{1}{3}\left(b-\frac{a}{3}\right)-\frac{2}{9}\left[a-\frac{3}{4}\left(b-\frac{4a}{5}\right)\right]\right\}.$

20. $35\left[\frac{3x-4y}{5}-\frac{1}{10}\{3x-\frac{5}{7}(7x-4y)\}\right]+8(y-2x).$

21. $\frac{3}{8}\{\frac{4}{3}(a-b)-8(b-c)\}-\left\{\frac{b-c}{2}-\frac{c-a}{3}\right\}-\frac{1}{2}\{c-a-\frac{2}{3}(a-b)\}.$

22. $\frac{1}{2}x-\frac{1}{2}(\frac{2}{3}y-\frac{1}{2}z)-[x-\{\frac{1}{2}x-(\frac{1}{3}y-\frac{1}{4}z)\}-(\frac{2}{3}y-\frac{1}{2}z)].$

INSERTION OF BRACKETS.

68. The converse operation of inserting brackets is important. The rules for doing this have been enunciated in Arts. 25, 26; for convenience we repeat them.

Rule I. *Any part of an expression may be enclosed within brackets and the sign + prefixed, the sign of every term within the brackets remaining unaltered.*

Ex. $a-b+c-d-e=a-b+(c-d-e).$

Rule II. *Any part of an expression may be enclosed within brackets and the sign − prefixed, provided the sign of every term within the brackets be changed.*

Ex. $a-b+c-d-e=a-(b-c)-(d+e).$

69. The terms of an expression can be bracketed in various ways.

Ex. The expression $ax-bx+cx-ay+by-cy$

may be written $(ax-bx)+(cx-ay)+(by-cy)$,

or $(ax-bx+cx)-(ay-by+cy)$,

or $(ax-ay)-(bx-by)+(cx-cy)$.

70. A factor, common to every term within a bracket, may be removed and placed outside as a multiplier of the expression within the bracket.

Ex. **1.** In the expression

$$ax^3-cx+7-dx^2+bx-c-dx^3+bx^2-2x$$

bracket together the powers of x so as to have the sign + before each bracket.

$$\text{The expression } = (ax^3 - dx^3) + (bx^2 - dx^2) + (bx - cx - 2x) + (7 - c)$$
$$= x^3(a - d) + x^2(b - d) + x(b - c - 2) + (7 - c)$$
$$= (a - d)x^3 + (b - d)x^2 + (b - c - 2)x + 7 - c.$$

In this last result the compound expressions $a - d$, $b - d$, $b - c - 2$ are regarded as the coefficients of x^3, x^2, and x respectively.

Ex. 2. In the expression $-a^2x - 7a + a^2y + 3 - 2x - ab$ bracket together the powers of a so as to have the sign $-$ before each bracket.

$$\text{The expression } = -(a^2x - a^2y) - (7a + ab) - (2x - 3)$$
$$= -a^2(x - y) - a(7 + b) - (2x - 3)$$
$$= -(x - y)a^2 - (7 + b)a - (2x - 3).$$

EXAMPLES VI. c.

In the following expressions bracket the powers of x so that the signs before all the brackets shall be positive:

1. $ax^4 + bx^2 + 5 + 2bx - 5x^2 + 2x^4 - 3x.$
2. $3bx^2 - 7 - 2x + ab + 5ax^3 + cx - 4x^2 - bx^3.$
3. $2 - 7x^3 + 5ax^2 - 2cx + 9ax^3 + 7x - 3x^2.$

In the following expressions bracket the powers of x so that the signs before all the brackets shall be negative:

4. $ax^2 + 5x^3 - a^2x^4 - 2bx^3 - 3x^2 - bx^4.$
5. $7x^3 - 3c^2x - abx^5 + 5ax + 7x^5 - abcx^3.$
6. $3b^2x^4 - bx - ax^4 - cx^4 - 5c^2x - 7x^4.$

Simplify the following expressions, and in each result regroup the terms according to powers of x:

7. $ax^3 - 2cx - [bx^2 - \{cx - dx - (bx^3 + 3cx^2)\} - (cx^2 - bx)].$
8. $5ax^3 - 7(bx - cx^2) - \{6bx^2 - (3ax^2 + 2ax) - 4cx^3\}.$
9. $ax^2 - 3\{-ax^3 + 3bx - 4[\frac{1}{6}cx^3 - \frac{2}{3}(ax - bx^2)]\}.$
10. $x^5 - 4bx^4 - \frac{1}{6}\left[12ax - 4\left\{3bx^4 - 9\left(\frac{cx}{2} - bx^5\right) - \frac{2}{3}ax^4\right\}\right].$

71. In certain cases of addition, multiplication, etc., of expressions which involve literal coefficients, the results may be more conveniently written by grouping the terms according to powers of some common letter.

Ex. 1. Add together $ax^3 - 2bx^2 + 3$, $bx - cx^3 - x^2$, and $x^3 - ax^2 + cx$.

$$\text{The sum } = ax^3 - 2bx^2 + 3 + bx - cx^3 - x^2 + x^3 - ax^2 + cx$$
$$= ax^3 - cx^3 + x^3 - ax^2 - 2bx^2 - x^2 + bx + cx + 3$$
$$= (a - c + 1)x^3 - (a + 2b + 1)x^2 + (b + c)x + 3.$$

Ex. 2. Multiply $ax^2 - 2bx + 3c$ by $px - q$.

The product $= (ax^2 - 2bx + 3c)(px - q)$

$= apx^3 - 2bpx^2 + 3cpx - aqx^2 + 2bqx - 3cq$

$= apx^3 - (2bp + aq)x^2 + (3cp + 2bq)x - 3cq.$

EXAMPLES VI. d.

Add together the following expressions, and in each case arrange the result according to powers of x:

1. $ax^3 - 2cx,\ bx^2 - cx^3,$ and $cx^2 - x.$
2. $x^2 - x - 1,\ ax^2 - bx^3,\ bx + x^3.$
3. $a^2x^3 - 5x,\ 2ax^2 - 5ax^3,\ 2x^3 - bx^2 - ax.$
4. $ax^2 + bx - c,\ qx - r - px^2,\ x^2 + 2x + 3.$
5. $px^3 - qx,\ qx^2 - px,\ q - x^3,\ px^2 + qx^3.$

Multiply together the following expressions, and in each case arrange the result according to powers of x:

6. $ax^2 + bx + 1$ and $cx + 2$.
7. $cx^2 - 2x + 3$ and $ax - b$.
8. $ax^2 - bx - c$ and $px + q$.
9. $2x^2 - 3x - 1$ and $bx + c$.
10. $ax^2 - 2bx + 3c$ and $x - 1$.
11. $px^2 - 2x - q$ and $ax - 3$.
12. $x^3 + ax^2 - bx - c$ and $x^3 - ax^2 - bx + c$.
13. $ax^3 - x^2 + 3x - b$ and $ax^3 + x^2 + 3x + b$.
14. $x^4 - ax^3 - bx^2 + cx + d$ and $x^4 + ax^3 - bx^2 - cx + d$.

MISCELLANEOUS EXAMPLES II.

1. Find value of $(a - b)^2 + (b - c)^2 + (a - b) + 2c^2$ when $a = 1$, $b = 2$, $c = -3$.

2. Find the sum of $2x$, $3x^2$, 5, $-3x^3$, -4, x, $-6x^2$, $8x^3$, arranging result in descending powers.

3. Diminish the sum of $b^3 + 7b^2 - 5$ and $4b^2 - 3b + 7$ by $11b^2 + 2$.

4. Show that $(1+x)^2(1+y^2) - (1+x^2)(1+y)^2 = 2(x-y)(1-xy)$.

5. Simplify $(a + b)(a + c) - (a - b)(a - c)$.

6. Subtract the sum of $3m^3 - 4m + 1$ and $m^2 - 3m$ from $4m^3 + 2m^2 - 7m$.

7. What expression must be taken from the sum of $a + 3b$, $4a^2 - 5a$, $b^2 + 2a$, $2a - 3b^2$ in order to produce $a^2 - b^2$?

8. Find value of $a^2 + (c+d)\left(\sqrt{\left(\frac{ad}{c}\right)} + d\right)$ when $a=2$, $c=9$, $d=8$.

9. Multiply $a^2 + (b - c)^2$ by $b + c + 1$.

10. Divide $343x^3 + 512y^3$ by $7x + 8y$.

11. If $a = 1$, $b = 2$, $c = 3$, $d = 4$, find the value of

$$a + [(b - c)(2d - b)] - \left(\frac{\sqrt{(2ad - b^2)}}{d} - 4c\right).$$

12. What number must be added to $2x^2 - 3xy^2 + 6$ to produce $7x^2 + \frac{2}{3}xy^2 - x^2y + 5$?

13. Simplify $(x - y) - \{3x - (x + y)\} + \{(2x - 3y) - (x - 2y)\}$.

14. Show that $a(a - 1)(a - 2)(a - 3) = (a^2 - 3a + 1)^2 - 1$.

15. $x^4 - 10x^2 + 9 \div x^2 - 2x - 3$.

16. Simplify $9a - (2b - c) + 2d - (5a + 3b) + 4c - 2d$, and find its value when $a = 7$, $b = -3$, $c = -4$.

17. Multiply $3a^2b - 4ab^3c + 2a^3b^2c^3$ by $-6a^2b^2c^3$, and divide the result by $3ab^2c^2$.

18. If $a = -1$, $b = 2$, $c = 0$, $d = 1$, find the value of

$$ad + ac - a^2 - cd + c^2 - a + 2c + a^2b + 2a^3.$$

19. Find the sum of $3a + 2b$, $-5c - 2d$, $3e + 5f$, $b - a + 2d$, $-2a - 3b + 5c - 2f$.

20. Subtract $ax^2 - 4$ from zero, and add the difference to the sum of $2x^3 - 5x$ and unity.

21. Multiply $\frac{1}{6}x^2 + \frac{1}{3}xy - \frac{1}{2}y^2$ by $\frac{1}{10}x^2 - \frac{1}{2}xy + \frac{2}{5}y^2$.

22. Divide $6a^4 - a^3b + 2a^2b^2 + 13ab^3 + 4b^4$ by $2a^2 - 3ab + 4b^2$.

23. Simplify $5x^4 - 8x^3 - (2x^2 - 7) - (x^4 + 5) + (3x^3 - x)$, and subtract the result from $4x^4 - x + 2$.

24. Simplify by removing brackets $5[x - 4\{x - 3(2x - \overline{3x + 2})\}]$.

25. If $a = 1$, $b = 2$, $c = 3$, and $d = 4$, find value of

$$(c + d)(a - cd) + \sqrt{\frac{ab(cd + bcd) + c^2}{bd(b + c + d - a)}}.$$

26. Express by means of symbols

(1) b's excess over c is greater than a by 7.

(2) Three times the sum of a and $2b$ is less by 5 than the product of b and c.

27. Simplify

$$3a^2 - (4a - b^2) - \{2a^2 - (3b - a^2) - \overline{2b - 3a}\} - \{5b - 7a - (c^2 - b^2)\}.$$

28. Find the continued product of

$$x^2 + xy + y^2,\ x^2 - xy + y^2,\ x^4 - x^2y^2 + y^4.$$

CHAPTER VII.

SIMPLE EQUATIONS.

72. An **equation** asserts that two expressions are equal, but we do not usually employ the word equation in so wide a sense.

Thus the statement $x + 3 + x = 2x + 3$, which is *always* true whatever value x may have, is called an **identical equation**, or an **identity**. The **sign of identity** frequently used is $\equiv$.

The parts of an equation to the right and left of the sign of equality are called **members** or **sides** of the equation, and are distinguished as the *right side* and *left side*.

73. Certain equations are only true for particular values of the symbols employed. Thus $3x = 6$ is only true when $x = 2$, and is called an **equation of condition**, or more usually an **equation**. Consequently an **identity** is an equation which is *always* true whatever be the values of the symbols involved; whereas an **equation**, in the ordinary use of the word, is only true for *particular* values of the symbols. In the above example $3x = 6$, the value 2 is said to **satisfy** the equation. The object of the present chapter is to explain how to treat an equation of the simplest kind in order to discover the value which satisfies it.

74. The letter whose value it is required to find is called the **unknown quantity**. The process of finding its value is called **solving the equation**. The value so found is called the **root** or the **solution** of the equation.

75. The solution of equations, and the operations subsidiary to it, form an extremely important part of Mathematics. All sorts of mathematical problems consist in the indirect determination of some quantity by means of its relations to other quantities which are known, and these relations are all expressed by means of equations. The operation in general of solving a problem in Mathematics, other than a transformation, is first, to express the conditions of the problem by means of one or more equations, and secondly, to solve these equations. For example, the problem which is expressed by the equation above given is the very simple question, "What is the number such that if multiplied by 3, the product is 6?" In the present chapter, it is the second of these two operations, the solution of an equation, that is considered.

76. An equation which involves the unknown quantity in the **first degree** is called a **simple equation.**

The process of solving a simple equation depends upon the following **axioms:**

1. If to equals we add equals, the sums are equal.
2. If from equals we take equals, the remainders are equal.
3. If equals are multiplied by equals, the products are equal.
4. If equals are divided by equals, the quotients are equal.

77. Consider the equation $7x = 14$.

It is required to find what numerical value x must have consistent with this statement.

Dividing both sides by 7, we get

$$x = 2 \quad . \quad . \quad . \quad . \quad . \quad \text{(Axiom 4).}$$

Similarly, if $$\frac{x}{2} = -6,$$

multiplying both sides by 2, we get

$$x = -12 \quad . \quad . \quad . \quad . \quad \text{(Axiom 3).}$$

Again, in the equation $7x - 2x - x = 23 + 15 - 10$, by collecting terms, we have $4x = 28$.

$$\therefore x = 7.$$

TRANSPOSITION OF TERMS.

78. To solve $3x - 8 = x + 12$.

Here the unknown quantity occurs on both sides of the equation. We can, however, **transpose** any term from one side to the other by simply *changing its sign.* This we proceed to show.

Subtract x from both sides of the equation, and we get

$$3x - x - 8 = 12 \quad . \quad . \quad . \quad \text{(Axiom 2).}$$

Adding 8 to both sides, we have

$$3x - x = 12 + 8 \quad . \quad . \quad \text{(Axiom 1).}$$

Thus we see that $+x$ has been removed from one side, and appears as $-x$ on the other; and -8 has been removed from one side and appears as $+8$ on the other.

It is evident that similar steps may be employed in all cases. Hence we may enunciate the following rule:

Rule. *Any term may be transposed from one side of the equation to the other by changing its sign.*

79. We may **change the sign of every term** in an equation; for this is equivalent to multiplying both sides by -1, which does not destroy the equality (Axiom 3).

Ex. Take the equation $\qquad -3x - 12 = x - 24.$

Multiplying both sides by -1, $\; 3x + 12 = -x + 24,$

which is the original equation with the sign of every term changed.

80. We can now give a general rule for solving a simple equation with one unknown quantity.

Rule. *Transpose all the terms containing the unknown quantity to one side of the equation, and the known quantities to the other. Collect the terms on each side; divide both sides by the coefficient of the unknown quantity, and the value required is obtained.*

Ex. 1. Solve $5(x - 3) - 7(6 - x) + 3 = 24 - 3(8 - x)$.

Removing brackets, $\; 5x - 15 - 42 + 7x + 3 = 24 - 24 + 3x;$

transposing, $\qquad 5x + 7x - 3x = 24 - 24 + 15 + 42 - 3;$

collecting terms, $\qquad 9x = 54.$

$$\therefore\ x = 6.$$

Ex. 2. Solve $5x-(4x-7)(3x-5)=6-3(4x-9)(x-1)$.

Simplifying, we have

$$5x-(12x^2-41x+35)=6-3(4x^2-13x+9),$$

and by removing brackets,

$$5x-12x^2+41x-35=6-12x^2+39x-27.$$

Erase the term $-12x^2$ on each side and transpose;

thus $\quad 5x+41x-39x=6-27+35;$

collecting terms, $\quad 7x=14.$

$$\therefore\ x=2.$$

Note. Since the $-$ sign before a bracket affects every term within it, in the first line of work of Ex. 2, we do not remove the brackets until we have formed the products.

Ex. 3. Solve $7x-5[x-\{7-6(x-3)\}]=3x+1$.

Removing brackets, we have

$$7x-5[x-\{7-6x+18\}]=3x+1,$$
$$7x-5[x-25+6x]=3x+1,$$
$$7x-5x+125-30x=3x+1;$$

transposing, $\quad 7x-5x-30x-3x=1-125;$

collecting terms, $\quad -31x=-124;$

$$\therefore\ x=4.$$

81. It is extremely useful for the beginner to acquire the habit of occasionally **verifying**, that is, proving the truth of his results. Proofs of this kind are interesting and convincing; and the habit of applying such tests tends to make the student self-reliant and confident in his own accuracy.

In the case of simple equations we have only to show that when we substitute the value of x in the two sides of the equation we obtain the same result.

Ex. To show that $x=2$ *satisfies* the equation

$5x-(4x-7)(3x-5)=6-3(4x-9)(x-1)$. Ex. 2, Art. 80.

When $x=2$, the left side $5x-(4x-7)(3x-5)=10-(8-7)(6-5)$

$$=10-1=9.$$

The right side $6-3(4x-9)(x-1)=6-3(8-9)(2-1)$

$$=6-3(-1)=9.$$

Thus, since these two results are the same, $x=2$ satisfies the equation.

EXAMPLES VII.

Solve the following equations:

1. $3x + 15 = x + 25$. **2.** $2x - 3 = 3x - 7$.

3. $3x + 4 = 5(x - 2)$. **4.** $2x + 3 = 16 - (2x - 3)$.

5. $8(x - 1) + 17(x - 3) = 4(4x - 9) + 4$.

6. $15(x - 1) + 4(x + 3) = 2(7 + x)$.

7. $5x - 6(x - 5) = 2(x + 5) + 5(x - 4)$.

8. $8(x - 3) - (6 - 2x) = 2(x + 2) - 5(5 - x)$.

9. $7(25 - x) - 2x = 2(3x - 25)$.

10. $3(169 - x) - (78 + x) = 29x$.

11. $5x - 17 + 3x - 5 = 6x - 7 - 8x + 115$.

12. $7x - 39 - 10x + 15 = 100 - 33x + 26$.

13. $118 - 65x - 123 = 15x + 35 - 120x$.

14. $157 - 21(x + 3) = 163 - 15(2x - 5)$.

15. $179 - 18(x - 10) = 158 - 3(x - 17)$.

16. $97 - 5(x + 20) = 111 - 8(x + 3)$.

17. $x - [3 + \{x - (3 + x)\}] = 5$.

18. $5x - (3x - 7) - \{4 - 2x - (6x - 3)\} = 10$.

19. $14x - (5x - 9) - \{4 - 3x - (2x - 3)\} = 30$.

20. $25x - 19 - [3 - \{4x - 5\}] = 3x - (6x - 5)$.

21. $(x + 1)(2x + 1) = (x + 3)(2x + 3) - 14$.

22. $(x + 1)^2 - (x^2 - 1) = x(2x + 1) - 2(x + 2)(x + 1) + 20$.

23. $2(x + 1)(x + 3) + 8 = (2x + 1)(x + 5)$.

24. $6(x^2 - 3x + 2) - 2(x^2 - 1) = 4(x + 1)(x + 2) - 24$.

25. $2(x - 4) - (x^2 + x - 20) = 4x^2 - (5x + 3)(x - 4) - 64$.

26. $(x + 15)(x - 3) - (x^2 - 6x + 9) = 30 - 15(x - 1)$.

27. $2x - 5\{3x - 7(4x - 9)\} = 66$.

28. $20(2 - x) + 3(x - 7) - 2[x + 9 - 3\{9 - 4(2 - x)\}] = 22$.

29. $x + 2 - [x - 8 - 2\{8 - 3(5 - x) - x\}] = 0$.

30. $3(5 - 6x) - 5[x - 5\{1 - 3(x - 5)\}] = 23$.

31. $(x + 1)(2x + 3) = 2(x + 1)^2 + 8$.

32. $3(x - 1)^2 - 3(x^2 - 1) = x - 15$.

33. $(3x + 1)(2x - 7) = 6(x - 3)^2 + 7$.

34. $x^2 - 8x + 25 = x(x - 4) - 25(x - 5) - 16$.

35. $x(x + 1) + (x + 1)(x + 2) = (x + 2)(x + 3) + x(x + 4) - 9$.

36. $2(x + 2)(x - 4) = x(2x + 1) - 21$.

37. $(x + 1)^2 + 2(x + 3)^2 = 3x(x + 2) + 35$.

38. $4(x + 5)^2 - (2x + 1)^2 = 3(x - 5) + 180$.

39. $84 + (x + 4)(x - 3)(x + 5) = (x + 1)(x + 2)(x + 3)$.

40. $(x + 1)(x + 2)(x + 6) = x^3 + 9x^2 + 4(7x - 1)$.

CHAPTER VIII.

SYMBOLICAL EXPRESSION.

82. In solving algebraic problems the chief difficulty of the beginner is to express the conditions of the question by means of symbols. A question proposed in algebraic symbols will frequently be found puzzling, when a similar arithmetical question would present no difficulty. Thus, the answer to the question "find a number greater than x by a" may not be self-evident to the beginner, who would of course readily answer an analogous arithmetical question, "find a number greater than 50 by 6." The process of addition which gives the answer in the second case supplies the necessary hint; and, just as the number which is greater than 50 by 6 is $50 + 6$, so the number which is greater than x by a is $x + a$.

83. The following examples will perhaps be the best introduction to the subject of this chapter. After the first we leave to the student the choice of arithmetical instances, should he find them necessary.

Ex. 1. By how much does x exceed 17?

Take a numerical instance; "by how much does 27 exceed 17?"

The answer obviously is 10, which is equal to $27 - 17$.

Hence the excess of x over 17 is $x - 17$.

Similarly the defect of x from 17 is $17 - x$.

Ex. 2. If x is one *part* of 45 the other part is $45 - x$.

Ex. 3. How far can a man walk in a hours at the rate of 4 miles an hour?

In 1 hour he walks 4 miles.

In a hours he walks a times as far, that is, $4a$ miles.

Ex. 4. A and B are playing for money; A begins with $\$p$ and B with q dimes: after B has won $\$x$, how many dimes has each?

What B has won A has lost,

$$\therefore \text{ A has } 10\,(p - x) \text{ dimes,}$$
$$\text{B has } q + 10x \text{ dimes.}$$

EXAMPLES VIII. a.

1. What must be added to x to make y?

2. By what must 3 be multiplied to make a?

3. What dividend gives b as the quotient when 5 is the divisor?

4. What is the defect of $2c$ from $3d$.

5. By how much does $3k$ exceed k?

6. If 100 be divided into two parts, and one part be x, what is the other?

7. What number is less than 20 by c?

8. What is the price in cents of a oranges at ten cents a dozen?

9. If the difference of two numbers be 11, and if the smaller be x, what is the greater?

10. If the sum of two numbers be c, and one of them is 20, what is the other?

11. What is the excess of 90 over x?

12. By how much does x exceed 30?

13. If 100 contains x 5 times, what is the value of x.

14. What is the cost in dollars of 40 books at x dimes each?

15. In x years a man will be 36 years old, what is his present age?

16. How old will a man be in a years if his present age is x years?

17. If x men take 5 days to reap a field, how long will one man take?

18. What value of x will make $5x$ equal to 20?

19. What is the price in dimes of 120 apples, when the cost of two dozen is x cents?

20. How many hours will it take to walk x miles at 4 miles an hour?

21. How far can I walk in x hours at the rate of y miles an hour?

22. How many miles is it between two places, if a train travelling p miles an hour takes 5 hours to perform the journey?

23. A man has a dollars and b dimes, how many cents has he?

24. If I spend x half-dollars out of a sum of $20, how many half-dollars have I left?

25. Out of a purse containing $\$a$ and b dimes a man spends c cents; express in cents the sum left.

26. By how much does $2x - 5$ exceed $x + 1$?

27. What number must be taken from $a - 2b$ to leave $a - 3b$?

28. If a bill is shared equally amongst x persons, and each pays four dimes, how many cents does the bill amount to?

29. If I give away c dimes out of a purse containing a dollars and b half-dollars, how many dimes have I left?

30. If I spend x quarters a week, how many dollars do I save out of a yearly income of $\$y$?

31. A bookshelf contains x Latin, y Greek, and z English books; if there are 100 books, how many are there in other languages?

32. I have x dollars in my purse, y dimes in one pocket, and z cents in another; if I give away a half-dollar, how many cents have I left?

33. In a class of x boys, y work at Classics, z at Mathematics, and the rest are idle; what is the excess of workers over idlers?

84. We add a few harder examples worked out in full.

Ex. **1.** What is the present age of a man who x years hence will be m times as old as his son now aged y years?

In x years the son's age will be $y + x$ years; hence the father's age will be $m(y + x)$ years; therefore *now* the father's age is $m(y + x) - x$ years.

Ex. **2.** Find the simple interest on $\$k$ in n years at f per cent.

$$\text{Interest on } \$100 \text{ for 1 year is } \$f,$$
$$\therefore \quad \ldots\ldots\ \$1 \quad \ldots\ldots\ \$\frac{f}{100},$$
$$\therefore \quad \ldots\ldots\ \$k \quad \ldots\ldots\ \$\frac{kf}{100},$$
$$\therefore \quad \text{Interest on } \$k \text{ for } n \text{ years is } \$\frac{nkf}{100}.$$

Ex. **3.** A room is x yards long, y feet broad, and a feet high; find how many square yards of carpet will be required for the floor, and how many square yards of paper for the walls.

(1) The area of the floor is $3xy$ square feet;

$\therefore$ the number of square yards of carpet required is $\frac{3xy}{9}=\frac{xy}{3}$.

(2) The perimeter of the room is $2(3x+y)$ feet;
$\therefore$ the area of the walls is $2a(3x+y)$ square feet;

$\therefore$ number of square yards of paper required is $\frac{2a(3x+y)}{9}$.

Ex. **4.** The digits of a number beginning from the left are a, b, c; what is the number?

Here c is the digit in the units' place; b standing in the tens' place represents b tens; similarly a represents a hundreds.

The number is therefore equal to a hundreds $+ b$ tens $+ c$ units

$$= 100a + 10b + c.$$

If the digits of the number are inverted, a new number is formed which is symbolically expressed by

$$100c + 10b + a.$$

Ex. **5.** What is (1) the sum, (2) the product of three consecutive numbers of which the least is n?

The numbers consecutive to n are $n+1$, $n+2$.

$$\therefore \text{ the sum} = n + (n+1) + (n+2)$$
$$= 3n + 3.$$

And the product $= n(n+1)(n+2)$.

We may remark here that any *even* number may be denoted by $2n$, where n is *any* positive whole number; for this expression is exactly divisible by 2.

Similarly, any odd number may be denoted by $2n+1$; for this expression when divided by 2 leaves remainder 1.

EXAMPLES VIII. b.

1. Write four consecutive numbers of which x is the least.

2. Write three consecutive numbers of which y is the greatest.

3. Write five consecutive numbers of which x is the middle one.

4. What is the next even number after $2n$?

5. What is the odd number next before $2x+1$?

6. Find the sum of three consecutive odd numbers of which the middle one is $2n+1$.

7. A man makes a journey of x miles. He travels a miles by coach, b by train, and finishes the journey by boat. How far does the boat carry him?

8. A horse eats a bushels and a donkey b bushels of corn in a week; how many bushels will they together consume in n weeks?

9. If a man was x years old 5 years ago, how old will he be y years hence?

10. A boy is x years old, and five years hence his age will be half that of his father. How old is the father now?

11. What is the age of a man who y years ago was m times as old as a child then aged x years?

12. A's age is double B's, B's is three times C's, and C is x years old: find A's age.

13. What is the interest on $1000 in b years at c per cent.?

14. What is the interest on $\$x$ in a years at 5 per cent.?

15. What is the interest on $\$50a$ in a years at a per cent.?

16. What is the interest on $\$24xy$ in x months at y per cent. per annum?

17. A room is x yards in length, and y feet in breadth; how many square feet are there in the area of the floor?

18. A square room measures x feet each way; how many square yards of carpet will be required to cover it?

19. A room is p feet long and x yards in width; how many yards of carpet two feet wide will be required for the floor?

20. What is cost in dollars of carpeting a room a yards long, b feet broad, with carpet costing c dimes a square yard?

21. A room is a yards long and b yards broad; in the middle there is a carpet c feet square; how many square yards of oil-cloth will be required to cover the rest of the floor?

22. How long will it take a person to walk b miles if he walks 20 miles in c hours?

23. A train is running with a velocity of x feet per second; how many miles will it travel in y hours?

24. How many men will be required to do in x hours what y men do in xz hours?

CHAPTER IX.

Problems Leading to Simple Equations.

85. The principles of the last chapter may now be employed to solve various problems.

The method of procedure is as follows:

Represent the unknown quantity by a symbol, as x, and express in symbolical language the conditions of the question; we thus obtain a simple equation which can be solved by the methods already given in Chapter VII.

Note. Unknown quantities are usually represented by the last letters of the alphabet.

Ex. 1. Find two numbers whose sum is 28, and whose difference is 4.

Let x represent the smaller number, then $x + 4$ represents the greater.

Their sum is $x + (x + 4)$, which is to be equal to 28.

Hence $$x + x + 4 = 28;$$
$$2x = 24;$$
$$\therefore \quad x = 12,$$
and $$x + 4 = 16,$$
so that the numbers are 12 and 16.

The beginner is advised to test his solution by proving that it *satisfies* the conditions of the question.

Ex. 2. Divide 60 into two parts, so that three times the greater may exceed 100 by as much as 8 times the less falls short of 200.

Let x represent the greater part, then $60 - x$ represents the less

Three times the greater part is $3x$, and its excess over 100 is
$$3x - 100.$$

Eight times the less is $8(60 - x)$, and its defect from 200 is
$$200 - 8(60 - x).$$

Whence the symbolical statement of the question is

$$3x - 100 = 200 - 8(60 - x);$$
$$3x - 100 = 200 - 480 + 8x,$$
$$480 - 100 - 200 = 8x - 3x,$$
$$5x = 180;$$
$$\therefore \quad x = 36, \text{ the greater part,}$$

and $$60 - x = 24, \text{ the less.}$$

Ex. **3**. Divide \$47 between A, B, C, so that A may have \$10 more than B, and B \$8 more than C.

Suppose that C has x dollars; then B has $x + 8$ dollars, and A has $x + 8 + 10$ dollars.

Hence $$x + (x + 8) + (x + 8 + 10) = 47;$$
$$x + x + 8 + x + 8 + 10 = 47,$$
$$3x = 21;$$
$$\therefore \quad x = 7,$$

so that C has \$7, B \$15, A \$25.

Ex. **4**. A person spent \$112.80 in buying geese and ducks; if each goose cost 14 dimes, and each duck 6 dimes, and if the total number of birds bought was 108, how many of each did he buy?

In questions of this kind it is of essential importance to have all quantities expressed in the same denomination; in the present instance it will be convenient to express the money in dimes.

Let x represent the number of geese, then $108 - x$ represents the number of ducks.

Since each goose cost 14 dimes, x geese cost $14x$ dimes.

And since each duck cost 6 dimes, $108 - x$ ducks cost $6(108 - x)$ dimes.

Therefore the amount spent is

$$14x + 6(108 - x) \text{ dimes;}$$

but the question states that the amount is also \$112.80, that is, 1128 dimes.

Hence $$14x + 6(108 - x) = 1128;$$

dividing by 2, $$7x + 324 - 3x = 564,$$
$$4x = 240;$$
$$\therefore \quad x = 60, \text{ the number of geese;}$$

and $$108 - x = 48, \text{ the number of ducks.}$$

Ex. 5. A is twice as old as B ; ten years ago he was four times as old ; what are their present ages ?

Let x represent B's age in years, then $2x$ represents A's age.

Ten years ago their ages were respectively, $x-10$ and $2x-10$ years ; thus we have $2x-10=4(x-10)$;

$$2x-10=4x-40,$$

$$2x=30;$$

$$\therefore\ x=15,$$

so that B is 15 years old, A 30 years.

NOTE. In the above examples the unknown quantity x represents a *number* of dollars, ducks, years, etc. ; and the student must be careful to avoid beginning a solution with a supposition of the kind, "let $x=$ A's share," or "let $x=$ the ducks," or any statement so vague and inexact.

EXAMPLES IX.

1. One number exceeds another by 5, and their sum is 29 ; find them.

2. The difference between two numbers is 8 ; if 2 be added to the greater the result will be three times the smaller ; find the numbers.

3. Find a number such that its excess over 50 may be greater by 11 than its defect from 89.

4. What number is that which exceeds 8 by as much as its double exceeds 20 ?

5. Find the number which multiplied by 4 exceeds 40 as much as 40 exceeds the original number.

6. A man walks 10 miles, then travels a certain distance by train, and then twice as far by coach. If the whole journey is 70 miles, how far does he travel by train ?

7. What two numbers are those whose sum is 58, and difference 28 ?

8. If 288 be added to a certain number, the result will be equal to three times the excess of the number over 12 ; find the number.

9. Twenty-three times a certain number is as much above 14 as 16 is above seven times the number ; find it.

10. Divide 105 into two parts, one of which diminished by 20 shall be equal to the other diminished by 15.

11. Divide 128 into two parts, one of which is three times as large as the other.

12. Find three consecutive numbers whose sum shall equal 84.

13. The difference of the squares of two consecutive numbers is 35; find them.

14. The sum of two numbers is 8, and one of them with 22 added to it is five times the other; find the numbers.

15. Find two numbers differing by 10 whose sum is equal to twice their difference.

16. A and B begin to play each with \$60. If they play till A's money is double B's, what does A win?

17. Find a number such that if 5, 15, and 35 are added to it, the product of the first and third results may be equal to the square of the second.

18. The difference between the squares of two consecutive numbers is 121; find the numbers.

19. The difference of two numbers is 3, and the difference of their squares is 27; find the numbers.

20. Divide \$380 between A, B, and C, so that B may have \$30 more than A, and C may have \$20 more than B.

21. A sum of \$7 is made up of 46 coins which are either quarters or dimes; how many are there of each?

22. If silk costs five times as much as linen, and I spend \$48 in buying 22 yards of silk and 50 yards of linen, find the cost of each per yard.

23. A father is four times as old as his son; in 24 years he will only be twice as old; find their ages.

24. A is 25 years older than B, and A's age is as much above 20 as B's is below 85; find their ages.

25. A's age is three times B's, and in 18 years A will be twice as old as B; find their ages.

26. A is four times as old as B, and in 20 years will be twice as old as C, who is 5 years older than B; find their ages.

27. A's age is six times B's, and fifteen years hence A will be three times as old as B; find their ages.

28. A sum of \$16 was paid in dollars, half-dollars, and dimes. The number of half-dollars used was four times the number of dollars and twice the number of dimes; how many were there of each?

29. The sum of the ages of A and B is 30 years, and five years hence A will be three times as old as B; find their present ages.

30. I spend \$69.30 in buying 20 yards of calico and 30 yards of silk; the silk costs as many quarters per yard as the calico costs cents per yard; find the price of each.

31. I purchase 127 bushels of grain. If the number of bushels of wheat be double that of the corn, and seven more than five times the number of bushels of corn equals the number of bushels of oats, find the number of bushels of each.

32. The length of a room exceeds its breadth by 3 feet; if the length had been increased by 3 feet, and the breadth diminished by 2 feet, the area would not have been altered; find the dimensions.

33. The length of a room exceeds its breadth by 8 feet; if each had been increased by 2 feet, the area would have been increased by 60 feet; find the original dimensions of the room.

34. A and B start from the same place walking at different rates; when A has walked 15 miles B doubles his pace, and 6 hours later passes A; if A walks at the rate of 5 miles an hour, what is B's rate at first?

35. A sum of money is divided among A, B, and C, so that A and B have together \$20, A and C \$30, and B and C \$40; find the share of each.

36. A man sold two pieces of cloth, losing \$6 more on the one than on the other. If his entire loss was \$4 less than four times the smaller loss, find the amount lost on each piece.

37. Two men received the same sum for their labor; but if one had received \$10 more, and the other \$8 less, then one would have had three times as much as the other. What did each receive?

38. In a certain examination the number of successful candidates was four times the number of those who failed. If there had been 14 more candidates and 6 less had failed, the number of those who passed would have been five times the number of those who failed. Find the number of candidates.

39. A purse contains 14 coins, some of which are quarters and the rest dimes. If the coins are worth \$2 altogether, how many are there of each kind?

40. An estate was divided among three persons in such a way that the share of the first was three times that of the second, and the share of the second twice that of the third. The first received \$900 more than the third. How much did each receive?

CHAPTER X.

Resolution into Factors.

86. Definition. When an algebraic expression is the product of two or more expressions, each of these latter quantities is called a **factor** of it, and the determination of these quantities is called the **resolution** of the expression into its factors.

87. **Rational expressions** do not contain square or other roots (Art. 14) in any term.

88. **Integral expressions** do not contain a *letter* in the denominator of any term. Thus, $x^2 + 3xy + 2y^2$, and $\frac{1}{3}x^2 + \frac{1}{2}xy - \frac{1}{2}y^2$ are integral expressions.

89. In this chapter we shall explain the principal rules by which the resolution of rational and integral expressions into their component factors, which are rational and integral expressions, may be effected.

WHEN EACH OF THE TERMS IS DIVISIBLE BY A COMMON FACTOR.

90. The expression may be simplified by dividing each term separately by this factor, and enclosing the quotient within brackets; the common factor being placed outside as a coefficient.

Ex. **1.** The terms of the expression $3a^2 - 6ab$ have a common factor $3a$.

$$\therefore\ 3a^2 - 6ab = 3a(a - 2b).$$

Ex. **2.** $5a^2bx^3 - 15abx^2 - 20b^3x^2 = 5bx^2(a^2x - 3a - 4b^2)$.

EXAMPLES X. a.

Resolve into factors:

1. $a^3 - ax$.	**5.** $8x - 2x^2$.	**9.** $15a^2 - 225a^4$.
2. $x^3 - x^2$.	**6.** $5ax - 5a^3x^2$.	**10.** $54 - 81x$.
3. $2a - 2a^2$.	**7.** $15 + 25x^2$.	**11.** $10x^3 - 25x^4y$.
4. $7p^2 + p$.	**8.** $16x + 64x^2y$.	**12.** $3x^3 - x^2 + x$.

13. $3a^4 - 3a^3b + 6a^2b^2$.

14. $2x^2y^3 - 6x^2y^2 + 2xy^3$.

15. $6x^3 - 9x^2y + 12xy^2$.

16. $5x^5 - 10a^2x^3 - 15a^3x^3$.

17. $7a - 7a^3 + 14a^4$.

18. $38a^3x^5 + 57a^4x^2$.

WHEN THE TERMS CAN BE GROUPED SO AS TO CONTAIN A COMMON FACTOR.

91. Ex. 1. Resolve into factors $x^2 - ax + bx - ab$.

Noticing that the first two terms contain a factor x, and the last two terms a factor b, we enclose the first two terms in one bracket, and the last two in another. Thus,

$$\begin{aligned} x^2 - ax + bx - ab &= (x^2 - ax) + (bx - ab) \\ &= x(x-a) + b(x-a) \quad . \quad . \quad . \quad (1) \\ &= (x-a)(x+b), \end{aligned}$$

since each bracket of (1) contains the same factor $x - a$.

Ex. 2. Resolve into factors $6x^2 - 9ax + 4bx - 6ab$.

$$\begin{aligned} 6x^2 - 9ax + 4bx - 6ab &= (6x^2 - 9ax) + (4bx - 6ab) \\ &= 3x(2x - 3a) + 2b(2x - 3a) \\ &= (2x - 3a)(3x + 2b). \end{aligned}$$

Ex. 3. Resolve into factors $12a^2 - 4ab - 3ax^2 + bx^2$.

$$\begin{aligned} 12a^2 - 4ab - 3ax^2 + bx^2 &= (12a^2 - 4ab) - (3ax^2 - bx^2) \\ &= 4a(3a - b) - x^2(3a - b) \\ &= (3a - b)(4a - x^2). \end{aligned}$$

Note. In the first line of work it is usually sufficient to see that each pair contains some common factor. Any suitably chosen pairs will bring out the same result. Thus, in the last example, by a different arrangement, we have

$$\begin{aligned} 12a^2 - 4ab - 3ax^2 + bx^2 &= (12a^2 - 3ax^2) - (4ab - bx^2) \\ &= 3a(4a - x^2) - b(4a - x^2) \\ &= (4a - x^2)(3a - b). \end{aligned}$$

The same result as before, for the order in which the factors of a product are written is of course immaterial.

EXAMPLES X. b.

Resolve into factors:

1. $a^2 + ab + ac + bc$.
2. $a^2 - ac + ab - bc$.
3. $a^2c^2 + acd + abc + bd$.
4. $a^2 + 3a + ac + 3c$.
5. $2x + cx + 2c + c^2$.
6. $x^2 - ax + 5x - 5a$.
7. $5a + ab + 5b + b^2$.
8. $ab - by - ay + y^2$.
9. $mx - my - nx + ny$.
10. $mx - ma + nx - na$.
11. $2ax + ay + 2bx + by$.
12. $3ax - bx - 3ay + by$.
13. $6x^2 + 3xy - 2ax - ay$.
14. $mx - 2my - nx + 2ny$.
15. $ax^2 - 3bxy - axy + 3by^2$.
16. $x^2 + mxy - 4xy - 4my^2$.
17. $2x^4 - x^3 + 4x - 2$.
18. $3x^3 + 5x^2 + 3x + 5$.
19. $x^4 + x^3 + 2x + 2$.
20. $y^3 - y^2 + y - 1$.
21. $axy + bcxy - az - bcz$.
22. $f^2x^2 + g^2x^2 - ag^2 - af^2$.

TRINOMIAL EXPRESSIONS.

92. When the Coefficient of the Highest Power is Unity. Before proceeding to the next case of resolution into factors the student is advised to refer to Chap. IV. Art. 51. Attention has there been drawn to the way in which, in forming the product of two binomials, the coefficients of the different terms combine so as to give a trinomial result. Thus, by Art. 51,

$$(x+5)(x+3) = x^2 + 8x + 15 \quad \ldots\ldots \quad (1),$$
$$(x-5)(x-3) = x^2 - 8x + 15 \quad \ldots\ldots \quad (2),$$
$$(x+5)(x-3) = x^2 + 2x - 15 \quad \ldots\ldots \quad (3),$$
$$(x-5)(x+3) = x^2 - 2x - 15 \quad \ldots\ldots \quad (4).$$

We now propose to consider the converse problem: namely, the resolution of a trinomial expression, similar to those which occur on the right-hand side of the above identities, into its component binomial factors.

By examining the above results, we notice that:

1. The first term of both the factors is x.

2. The *product* of the second terms of the two factors is equal to the *third term* of the trinomial; thus in (2) above we see that 15 is the product of -5 and -3; while in (3) -15 is the product of $+5$ and -3.

3. The *algebraic sum* of the second terms of the two factors is equal to the *coefficient of* x in the trinomial; thus in (4) the sum of -5 and $+3$ gives -2, the coefficient of x in the trinomial.

In showing the application of these laws we will first consider a case where the *third term of the trinomial is positive.*

Ex. 1. Resolve into factors $x^2 + 11x + 24$.

The second terms of the factors must be such that their product is $+24$, and their sum $+11$. It is clear that they must be $+8$ and $+3$.

$$\therefore\ x^2 + 11x + 24 = (x+8)(x+3).$$

Ex. 2. Resolve into factors $x^2 - 10x + 24$.

The second terms of the factors must be such that their product is $+24$, and their sum -10. Hence they must *both* be *negative*, and it is easy to see that they must be -6 and -4.

$$\therefore\ x^2 - 10x + 24 = (x-6)(x-4).$$

Ex. 3. $\quad x^2 - 18x + 81 = (x-9)(x-9) = (x-9)^2.$

Ex. 4. $\quad x^4 + 10x^2 + 25 = (x^2+5)(x^2+5) = (x^2+5)^2.$

Ex. 5. Resolve into factors $x^2 - 11ax + 10a^2$.

The second terms of the factors must be such that their product is $+10a^2$, and their sum $-11a$. Hence they must be $-10a$ and $-a$.

$$\therefore\ x^2 - 11ax + 10a^2 = (x-10a)(x-a).$$

Note. In examples of this kind the student should always verify his results, by forming the product (*mentally*, as explained in Chapter IV.) of the factors he has chosen.

EXAMPLES X. c.

Resolve into factors:

1. $a^2 + 3a + 2$.
2. $a^2 + 2a + 1$.
3. $a^2 + 7a + 12$.
4. $a^2 - 7a + 12$.
5. $x^2 - 11x + 30$.
6. $x^2 - 15x + 56$.
7. $x^2 - 21x + 108$.
8. $x^2 - 21x + 80$.
9. $x^2 + 21x + 90$.
10. $x^2 - 19x + 84$.
11. $x^2 - 19x + 78$.
12. $x^2 - 18x + 45$.
13. $x^2 + 20x + 96$.
14. $x^2 - 26x + 165$.
15. $x^2 - 21x + 104$.
16. $a^2 + 30a + 225$.
17. $a^2 + 54a + 729$.
18. $a^2 - 38a + 361$.
19. $a^2 - 14ab + 49b^2$.
20. $a^2 + 5ab + 6b^2$.
21. $m^2 - 13mn + 40n^2$.
22. $m^2 - 22mn + 105n^2$.
23. $x^2 - 23xy + 132y^2$.
24. $x^2 - 26xy + 169y^2$.
25. $x^4 + 8x^2 + 7$.
26. $x^4 + 9x^2y^2 + 14y^4$.

27. $x^2 + 49\,xy + 600\,y^2$.
28. $x^2y^2 + 34\,xy + 289$.
29. $a^4b^4 + 37\,a^2b^2 + 300$.
30. $a^2 - 29\,ab + 54\,b^2$.
31. $x^4 + 162\,x^2 + 6561$.
32. $12 - 7\,x + x^2$.
33. $20 + 9\,x + x^2$.
34. $132 - 23\,x + x^2$.
35. $88 + 19\,x + x^2$.
36. $130 + 31\,xy + x^2y^2$.
37. $204 - 29\,x^2 + x^4$.
38. $216 + 35\,x + x^2$.

93. Next consider a case where *the third term of the trinomial is negative.*

Ex. 1. Resolve into factors $x^2 + 2\,x - 35$.

The second terms of the factors must be such that their product is -35, and their *algebraic sum* $+2$. Hence they must have *opposite* signs, and the greater of them must be *positive* in order to give its sign to their sum.

The required terms are therefore $+7$ and -5.

$$\therefore\ x^2 + 2\,x - 35 = (x + 7)(x - 5).$$

Ex. 2. Resolve into factors $x^2 - 3\,x - 54$.

The second terms of the factors must be such that their product is -54, and their *algebraic sum* -3. Hence they must have *opposite* signs, and the greater of them must be *negative* in order to give its sign to their sum.

The required terms are therefore -9 and $+6$.

$$\therefore\ x^2 - 3\,x - 54 = (x - 9)(x + 6).$$

Remembering that in these cases the numerical quantities *must have opposite signs*, if preferred, the following method may be adopted.

Ex. 3. Resolve into factors $x^2y^2 + 23\,xy - 420$.

Find two numbers whose product is 420, and whose *difference* is 23. These are 35 and 12; hence inserting the signs so that the positive may predominate, we have

$$x^2y^2 + 23\,xy - 420 = (xy + 35)(xy - 12).$$

EXAMPLES X. d.

Resolve into factors:

1. $x^2 - x - 2$.
2. $x^2 + x - 2$.
3. $x^2 - x - 6$.
4. $x^2 - 2\,x - 3$.
5. $x^2 + 2\,x - 3$.
6. $x^2 + x - 56$.
7. $x^2 - 4\,x - 12$.
8. $a^2 - a - 20$.
9. $a^2 + a - 20$.
10. $a^2 - 4\,a - 117$.
11. $x^2 + 9\,x - 36$.
12. $x^2 + x - 156$.

13. $x^2 + x - 110$.
14. $x^2 - 9x - 90$.
15. $x^2 - x - 240$.
16. $a^2 - 12a - 85$.
17. $a^2 - 11a - 152$.
18. $x^2y^2 - 5xy - 24$.
19. $x^2 + ax - 42a^2$.
20. $x^2 - 32xy - 105y^2$.
21. $x^2 + 18x - 115$.
22. $x^2 + 16x - 260$.
23. $a^2 - 11a - 26$.
24. $a^2y^2 + 14ay - 240$.
25. $a^4 - a^2b^2 - 56b^4$.
26. $x^4 - 14x^2 - 51$.
27. $y^4 + 6x^2y^2 - 27x^4$.
28. $a^2 + 12abx - 28b^2x^2$.
29. $a^2 - 18axy - 243x^2y^2$.
30. $x^4 + 13a^2x^2 - 300a^4$.
31. $x^4 - a^2x^2 - 132a^4$.
32. $x^4 - a^2x^2 - 462a^4$.
33. $x^6 + x^3 - 870$.
34. $2 + x - x^2$.
35. $6 + x - x^2$.
36. $110 - x - x^2$.
37. $380 - x - x^2$.
38. $120 - 7ax - a^2x^2$.

94. When the Coefficient of the Highest Power is not Unity.

Again, referring to Chap. IV. Art. 51, we may write the following results:

$$(3x + 2)(x + 4) = 3x^2 + 14x + 8 \quad . \quad . \quad . \quad (1),$$
$$(3x - 2)(x - 4) = 3x^2 - 14x + 8 \quad . \quad . \quad . \quad (2),$$
$$(3x + 2)(x - 4) = 3x^2 - 10x - 8 \quad . \quad . \quad . \quad (3),$$
$$(3x - 2)(x + 4) = 3x^2 + 10x - 8 \quad . \quad . \quad . \quad (4).$$

The converse problem presents more difficulty than the cases we have yet considered.

Consider the result $3x^2 - 14x + 8 = (3x - 2)(x - 4)$.

The first term $3x^2$ is the product of $3x$ and x.

The third term $+8$ is the product of -2 and -4.

The middle term $-14x$ is the result of adding together the two products $3x \times -4$ and $x \times -2$.

Again, consider the result $3x^2 - 10x - 8 = (3x + 2)(x - 4)$.

The first term $3x^2$ is the product of $3x$ and x.

The third term -8 is the product of $+2$ and -4.

The middle term $-10x$ is the result of adding together the two products $3x \times -4$ and $x \times 2$; and its sign is negative because the greater of these two products is negative.

Considering in a similar manner results (1) and (4), we see that:

1. If the *third term* of the trinomial is *positive*, then the *second terms* of its factors have both the *same sign*, and this sign is the same as that of the *middle term* of the trinomial.

2. If the *third term* of the trinomial is *negative*, then the *second terms* of its factors have *opposite signs.*

95. The beginner will frequently find that it is not easy to select the proper factors at the first trial. Practice alone will enable him to detect at a glance whether any pair he has chosen will combine so as to give the correct coefficients of the expression to be resolved.

Ex. Resolve into factors $7x^2 - 19x - 6$.

Write $(7x \quad 3)\,(x \quad 2)$ for a first trial, noticing that 3 and 2 must have opposite signs. These factors give $7x^2$ and -6 for the first and third terms. But since $7 \times 2 - 3 \times 1 = 11$, the combination fails to give the correct coefficient of the middle term.

Next try $(7x \quad 2)\,(x \quad 3)$.

Since $7 \times 3 - 2 \times 1 = 19$, these factors will be correct if we insert the signs so that the negative shall predominate.

Thus $$7x^2 - 19x - 6 = (7x + 2)(x - 3).$$

[Verify by mental multiplication.]

96. In actual work it will not be necessary to put down all these steps at length. The student will soon find that the different cases may be rapidly reviewed, and the unsuitable combinations rejected at once.

Ex. 1. Resolve into factors $14x^2 + 29x - 15$ . . . . (1),

$14x^2 - 29x - 15$ . . . . (2).

In each case we may write $(7x \quad 3)\,(2x \quad 5)$ as a first trial, noticing that 3 and 5 must have opposite signs.

And since $7 \times 5 - 3 \times 2 = 29$, we have only now to insert the proper signs in each factor.

In (1) the positive sign must predominate.

In (2) the negative sign must predominate.

Therefore $$14x^2 + 29x - 15 = (7x - 3)(2x + 5).$$
$$14x^2 - 29x - 15 = (7x + 3)(2x - 5).$$

Ex. 2. Resolve into factors $5x^2 + 17x + 6$ . . . . (1),

$5x^2 - 17x + 6$ . . . . (2).

In (1) we notice that the factors which give 6 are both positive.

In (2) we notice that the factors which give 6 are both negative.

And therefore for (1) we may write $(5x+\quad)(x+\quad)$.
(2) we may write $(5x-\quad)(x-\quad)$.

And, since $5\times 3+1\times 2=17$, we see that

$$5x^2+17x+6=(5x+2)(x+3).$$
$$5x^2-17x+6=(5x-2)(x-3).$$

In each expression the third term 6 also admits of factors 6 and 1, but this is one of the cases referred to above which the student would reject at once as unsuitable.

Ex. 3. $$9x^2-48xy+64y^2=(3x-8y)(3x-8y)$$
$$=(3x-8y)^2.$$

Ex. 4. $$6+7x-5x^2=(3+5x)(2-x).$$

NOTE. In Chapter XXVI. a method of obtaining the factors of *any* trinomial in the form ax^2+bx+c is given.

EXAMPLES X. e.

Resolve into factors:

1. $2x^2+3x+1$.
2. $3x^2+5x+2$.
3. $2x^2+5x+2$.
4. $3x^2+10x+3$.
5. $2x^2+9x+4$.
6. $3x^2+8x+4$.
7. $2x^2+11x+5$.
8. $3x^2+11x+6$.
9. $5x^2+11x+2$.
10. $3x^2+x-2$.
11. $4x^2+11x-3$.
12. $3x^2+14x-5$.
13. $2x^2+15x-8$.
14. $2x^2-x-1$.
15. $3x^2+7x-6$.
16. $2x^2+x-28$.
17. $3x^2+13x-30$.
18. $6x^2+7x-3$.
19. $2x^2-x-15$.
20. $3x^2+19x-14$.
21. $6x^2-31x+35$.
22. $4x^2+x-14$.
23. $3x^2-13x+14$.
24. $4x^2+23x+15$.
25. $2x^2-5xy-3y^2$.
26. $8x^2-38x+35$.
27. $15x^2-77x+10$.
28. $12x^2-31x-15$.
29. $24x^2+22x-21$.
30. $72x^2-145x+72$.
31. $24x^2-29xy-4y^2$.
32. $2-3x-2x^2$.
33. $6+5x-6x^2$.
34. $4-5x-6x^2$.
35. $5+32x-21x^2$.
36. $18-33x+5x^2$.
37. $8+6x-5x^2$.
38. $20-9x-20x^2$.
39. $10-5x-15x^2$.

97. We add an exercise containing miscellaneous examples on the preceding cases.

EXAMPLES X. f.

Resolve into factors:

1. $x^2+13x+42$.
2. $143-24ax+a^2x^2$.
3. $2x^2+7x+6$.
4. $a^2b^2-3abc-10c^2$.

5. $x^2 + x - 6$.

6. $2ax^2 + 3axy - 2bxy - 3by^2$.

7. $x^2 + 7xy - 60y^2$.

8. $a^2 - ay - 210y^2$.

9. $x^2 - 21x + 110$.

10. $24 + 37x - 72x^2$.

11. $98 - 7x - x^2$.

12. $3x^2 + 23x + 14$.

13. $2x^2 + 3x - 2$.

14. $x^2 - 20xy - 96y^2$.

15. $a^2 - 20abx + 75b^2x^2$.

16. $a^2 - 24a + 95$.

17. $7 + 10x + 3x^2$.

18. $a^2 - 4a - 21$.

19. $x^2 + 43xy + 390y^2$.

20. $x^2 + 23x + 102$.

21. $amx^2 + bmxy - anxy - bny^2$.

22. $6x^2 - 7x - 3$.

23. $3 + 11x - 4x^2$.

24. $12x^2 - 23xy + 10y^2$.

25. $3x^2 + 7x + 4$.

26. $a^2 - 32a + 256$.

27. $3x^2 - 19x - 14$.

28. $x^2 - 19x + 90$.

29. $x^2 + 3x - 40$.

30. $x^2y^2 - 16xy + 39$.

31. $204 - 5x - x^2$.

32. $15x^2 + 224x - 15$.

33. $3x^2 + 41x + 26$.

34. $65 + 8xy - x^2y^2$.

WHEN AN EXPRESSION IS THE DIFFERENCE OF TWO SQUARES.

98. By multiplying $a + b$ by $a - b$ we obtain the identity

$$(a + b)(a - b) = a^2 - b^2,$$

a result which may be verbally expressed as follows:

The product of the sum and the difference of any two quantities is equal to the difference of their squares.

Conversely, *the difference of the squares of any two quantities is equal to the product of the sum and the difference of the two quantities.*

Ex. 1. Resolve into factors $25x^2 - 16y^2$.

$$25x^2 - 16y^2 = (5x)^2 - (4y)^2.$$

Therefore the first factor is the sum of $5x$ and $4y$, and the second factor is the difference of $5x$ and $4y$.

$$\therefore 25x^2 - 16y^2 = (5x + 4y)(5x - 4y).$$

The intermediate steps may usually be omitted.

Ex. 2. $$1 - 49c^6 = (1 + 7c^3)(1 - 7c^3).$$

The difference of the squares of two numerical quantities is sometimes conveniently found by the aid of the formula

$$a^2 - b^2 = (a+b)(a-b).$$

Ex. $(329)^2 - (171)^2 = (329 + 171)(329 - 171)$
$= 500 \times 158 = 79000.$

EXAMPLES X. g.

Resolve into factors:

1. $x^2 - 4.$ **4.** $c^2 - 144.$ **7.** $400 - a^2.$ **10.** $36x^2 - 25b^2.$
2. $a^2 - 81.$ **5.** $49 - c^2.$ **8.** $x^2 - 9a^2.$ **11.** $9x^2 - 1.$
3. $y^2 - 100.$ **6.** $121 - x^2.$ **9.** $y^2 - 25x^2.$ **12.** $36p^2 - 49q^2.$

13. $4k^2 - 1.$ **24.** $1 - 36a^6.$ **35.** $x^2 - 25y^2.$
14. $49 - 100k^2.$ **25.** $9x^4 - a^2.$ **36.** $25 - 64x^2.$
15. $1 - 25x^2.$ **26.** $81x^6 - 25a^2.$ **37.** $121a^2 - 81x^2.$
16. $9x^2 - y^2.$ **27.** $x^4a^2 - 49.$ **38.** $p^2q^2 - 64a^4.$
17. $p^2q^2 - 36.$ **28.** $a^2 - 64x^6.$ **39.** $64x^2 - 25z^6.$
18. $a^2b^2 - 4c^2d^2.$ **29.** $a^2b^2 - 9x^6.$ **40.** $49x^4 - 16y^4.$
19. $x^4 - 9.$ **30.** $x^6y^6 - 4.$ **41.** $81p^4z^6 - 25b^2.$
20. $9a^4 - 121.$ **31.** $1 - a^2b^2.$ **42.** $16x^{16} - 9y^6.$
21. $25x^2 - 64.$ **32.** $9 - 4a^2.$ **43.** $36x^{36} - 49a^{14}.$
22. $81a^4 - 49x^4.$ **33.** $9a^4 - 25b^4.$ **44.** $1 - 100a^6b^4c^2.$
23. $x^6 - 25.$ **34.** $x^4 - 16b^2.$ **45.** $25x^{10} - 16a^8.$

Find, by resolving into factors, the value of:

46. $(575)^2 - (425)^2.$ **49.** $(339)^2 - (319)^2.$ **52.** $(1723)^2 - (277)^2.$
47. $(121)^2 - (120)^2.$ **50.** $(753)^2 - (253)^2.$ **53.** $(1639)^2 - (739)^2.$
48. $(750)^2 - (250)^2.$ **51.** $(101)^2 - (99)^2.$ **54.** $(1811)^2 - (689)^2.$

99. When One or Both of the Squares is a Compound Expression. We employ the method of the preceding articles, as is shown in the following examples:

Ex. 1. Resolve into factors $(a + 2b)^2 - 16x^2.$

The sum of $a + 2b$ and $4x$ is $a + 2b + 4x$, and their difference is $a + 2b - 4x.$

$$\therefore\ (a + 2b)^2 - 16x^2 = (a + 2b + 4x)(a + 2b - 4x).$$

Ex. 2. Resolve into factors $x^2-(2b-3c)^2$.

The sum of x and $2b-3c$ is $x+2b-3c$, and their difference is

$$x-(2b-3c)=x-2b+3c.$$

$$\therefore\ x^2-(2b-3c)^2=(x+2b-3c)(x-2b+3c).$$

If the factors contain like terms, they should be collected so as to give the result in its simplest form.

Ex. 3. $(3x+7y)^2-(2x-3y)^2$

$$=\{(3x+7y)+(2x-3y)\}\{(3x+7y)-(2x-3y)\}$$
$$=(3x+7y+2x-3y)(3x+7y-2x+3y)$$
$$=(5x+4y)(x+10y).$$

EXAMPLES X. h.

Resolve into factors:

1. $(a+b)^2-c^2$.
2. $(a-b)^2-c^2$.
3. $(x+y)^2-4z^2$.
4. $(x+2y)^2-a^2$.
5. $(a+3b)^2-16x^2$.
6. $(x+5a)^2-9y^2$.
7. $(x+5c)^2-1$.
8. $(a-2x)^2-b^2$.
9. $(2x-3a)^2-9c^2$.
10. $9x^2-(2a-3b)^2$.
11. $1-(a-b)^2$.
12. $c^2-(5a-3b)^2$.
13. $(a+b)^2-(c+d)^2$.
14. $(a-b)^2-(x+y)^2$.
15. $(7x+y)^2-1$.
16. $(a+b)^2-(m-n)^2$.
17. $(a-n)^2-(b+m)^2$.
18. $(b-c)^2-(a-x)^2$.
19. $(4a+x)^2-(b+y)^2$.
20. $(a+2b)^2-(3x+4y)^2$.
21. $1-(7a-3b)^2$.
22. $(a-b)^2-(x-y)^2$.
23. $(a-3x)^2-16y^2$.
24. $(2a-5x)^2-1$.
25. $(a+b-c)^2-(x-y+z)^2$.

Resolve into factors and simplify:

26. $(x+y)^2-x^2$.
27. $x^2-(y-x)^2$.
28. $(x+3y)^2-4y^2$.
29. $(24x+y)^2-(23x-y)^2$.
30. $(5x+2y)^2-(3x-y)^2$.
31. $9x^2-(3x-5y)^2$.
32. $(7x+3)^2-(5x-4)^2$.
33. $(3a+1)^2-(2a-1)^2$.
34. $16a^2-(3a+1)^2$.
35. $(2a+b-c)^2-(a-b+c)^2$.
36. $(x-7y+z)^2-(7y-z)^2$.
37. $(x+y-8)^2-(x-8)^2$.
38. $(2x+a-3)^2-(3-2x)^2$.

100. Compound Expressions Arranged as the Difference of two Squares. By suitably grouping the terms, compound expressions can often be expressed as the difference of two squares, and so be resolved into factors.

Ex. 1. Resolve into factors $a^2 - 2ax + x^2 - 4b^2$.

$$\begin{aligned} a^2 - 2ax + x^2 - 4b^2 &= (a^2 - 2ax + x^2) - 4b^2 \\ &= (a-x)^2 - (2b)^2 \\ &= (a - x + 2b)(a - x - 2b). \end{aligned}$$

Ex. 2. Resolve into factors $9a^2 - c^2 + 4cx - 4x^2$.

$$\begin{aligned} 9a^2 - c^2 + 4cx - 4x^2 &= 9a^2 - (c^2 - 4cx + 4x^2) \\ &= (3a)^2 - (c - 2x)^2 \\ &= (3a + c - 2x)(3a - c + 2x). \end{aligned}$$

Ex. 3. Resolve into factors $12xy + 25 - 4x^2 - 9y^2$.

$$\begin{aligned} 12xy + 25 - 4x^2 - 9y^2 &= 25 - (4x^2 - 12xy + 9y^2) \\ &= (5)^2 - (2x - 3y)^2 \\ &= (5 + 2x - 3y)(5 - 2x + 3y). \end{aligned}$$

Ex. 4. Resolve into factors $2bd - a^2 - c^2 + b^2 + d^2 + 2ac$.

Here the terms $2bd$ and $2ac$ suggest the proper preliminary arrangement of the expression. Thus

$$\begin{aligned} 2bd - a^2 - c^2 + b^2 + d^2 + 2ac &= b^2 + 2bd + d^2 - a^2 + 2ac - c^2 \\ &= b^2 + 2bd + d^2 - (a^2 - 2ac + c^2) \\ &= (b+d)^2 - (a-c)^2 \\ &= (b + d + a - c)(b + d - a + c). \end{aligned}$$

EXAMPLES X. k.

Resolve into factors:

1. $x^2 + 2xy + y^2 - a^2$.
2. $a^2 - 2ab + b^2 - x^2$.
3. $x^2 - 6ax + 9a^2 - 16b^2$.
4. $4a^2 + 4ab + b^2 - 9c^2$.
5. $x^2 + a^2 + 2ax - y^2$.
6. $2ay + a^2 + y^2 - x^2$.
7. $x^2 - a^2 - 2ab - b^2$.
8. $y^2 - c^2 + 2cx - x^2$.
9. $1 - x^2 - 2xy - y^2$.
10. $c^2 - x^2 - y^2 + 2xy$.
11. $x^2 + y^2 + 2xy - 4x^2y^2$.
12. $a^2 - 4ab + 4b^2 - 9a^2c^2$.
13. $x^2 + 2xy + y^2 - a^2 - 2ab - b^2$.
14. $a^2 - 2ab + b^2 - c^2 - 2cd - d^2$.
15. $x^2 - 4ax + 4a^2 - b^2 + 2by - y^2$.
16. $y^2 + 2by + b^2 - a^2 - 6ax - 9x^2$.
17. $x^2 - 2x + 1 - a^2 - 4ab - 4b^2$.
18. $9a^2 - 6a + 1 - x^2 - 8dx - 16d^2$.
19. $x^2 - a^2 + y^2 - b^2 - 2xy + 2ab$.
20. $a^2 + b^2 - 2ab - c^2 - d^2 - 2cd$.
21. $4x^2 - 12ax - c^2 - k^2 - 2ck + 9a^2$.
22. $a^2 + 6bx - 9b^2x^2 - 10ab - 1 + 25b^2$.
23. $a^4 - 25x^6 + 8a^2x^2 - 9 + 30x^3 + 16x^4$.

101. Important Cases. By a slight modification some expressions admit of being written in the form of the difference of two squares, and may then be resolved into factors by the method of Art. 98.

Ex. 1. Resolve into factors $x^4 + x^2y^2 + y^4$.

$$\begin{aligned} x^4 + x^2y^2 + y^4 &= (x^4 + 2x^2y^2 + y^4) - x^2y^2 \\ &= (x^2 + y^2)^2 - (xy)^2 \\ &= (x^2 + y^2 + xy)(x^2 + y^2 - xy) \\ &= (x^2 + xy + y^2)(x^2 - xy + y^2) \end{aligned}$$

Ex. 2. Resolve into factors $x^4 - 15x^2y^2 + 9y^4$.

$$\begin{aligned} x^4 - 15x^2y^2 + 9y^4 &= (x^4 - 6x^2y^2 + 9y^4) - 9x^2y^2 \\ &= (x^2 - 3y^2)^2 - (3xy)^2 \\ &= (x^2 - 3y^2 + 3xy)(x^2 - 3y^2 - 3xy). \end{aligned}$$

EXAMPLES X. 1.

Resolve into factors:

1. $x^4 + 16x^2 + 256$.
2. $81a^4 + 9a^2b^2 + b^4$.
3. $x^4 + y^4 - 7x^2y^2$.
4. $m^4 + n^4 - 18m^2n^2$.
5. $x^4 - 6x^2y^2 + y^4$.
6. $4x^4 + 9y^4 - 93x^2y^2$.
7. $4m^4 + 9n^4 - 24m^2n^2$.
8. $9x^4 + 4y^4 + 11x^2y^2$.
9. $x^4 - 19x^2y^2 + 25y^4$.
10. $16a^4 + b^4 - 28a^2b^2$.

WHEN AN EXPRESSION IS THE SUM OR DIFFERENCE OF TWO CUBES.

102. If we divide $a^3 + b^3$ by $a + b$ the quotient is $a^2 - ab + b^2$; and if we divide $a^3 - b^3$ by $a - b$ the quotient is $a^2 + ab + b^2$.

We have therefore the following identities:

$$a^3 + b^3 = (a + b)(a^2 - ab + b^2);$$
$$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

These results are very important, and enable us to resolve into factors any expression which can be written as the sum or the difference of two cubes.

Ex. 1.
$$\begin{aligned} 8x^3 - 27y^3 &= (2x)^3 - (3y)^3 \\ &= (2x - 3y)(4x^2 + 6xy + 9y^2). \end{aligned}$$

NOTE. The middle term $6xy$ is the *product* of $2x$ and $3y$.

Ex. 2. $$64a^3 + 1 = (4a)^3 + (1)^3$$
$$= (4a+1)(16a^2 - 4a + 1).$$

We may usually omit the intermediate step and write the factors at once.

Ex. 3. $343a^6 - 27x^3 = (7a^2 - 3x)(49a^4 + 21a^2x + 9x^2)$.

Ex. 4. $8x^9 + 729 = (2x^3 + 9)(4x^6 - 18x^3 + 81)$.

EXAMPLES X. m.

Resolve into factors:

1. $x^3 - y^3$.	**5.** $8x^3 - y^3$.	**9.** $a^3b^3 - c^3$.	**13.** $125 + a^3$.
2. $x^3 + y^3$.	**6.** $x^3 + 8y^3$.	**10.** $8x^3 + 27y^3$.	**14.** $216 - a^3$.
3. $x^3 - 1$.	**7.** $27x^3 + 1$.	**11.** $1 - 343x^3$.	**15.** $a^3b^3 + 512$.
4. $1 + a^3$.	**8.** $1 - 8y^3$.	**12.** $64 + y^3$.	**16.** $1000y^3 - 1$.

17. $x^3 + 64y^3$.	**25.** $x^3y^3 + z^3$.	**33.** $8x^3 - z^6$.
18. $27 - 1000x^3$.	**26.** $a^3b^3c^3 - 1$.	**34.** $216x^6 - b^3$.
19. $a^3b^3 + 216c^3$.	**27.** $343x^3 + 1000y^3$.	**35.** $a^3 + 343b^3$.
20. $343 - 8x^3$.	**28.** $729a^3 - 64b^3$.	**36.** $a^6 + 729b^3$.
21. $a^3 + 27b^3$.	**29.** $8a^3b^3 + 125x^3$.	**37.** $8x^3 - 729y^6$.
22. $27x^3 - 64y^3$.	**30.** $x^3y^3 - 216z^3$.	**38.** $p^3q^3 - 27x^3$.
23. $125x^3 - 1$.	**31.** $x^6 - 27y^3$.	**39.** $z^3 - 64y^6$.
24. $216p^3 - 343$.	**32.** $64x^6 + 125y^3$.	**40.** $x^3y^3 - 512$.

103. Miscellaneous Cases of Resolution into Factors.

Ex. 1. Resolve into factors $16a^4 - 81b^4$.

$$16a^4 - 81b^4 = (4a^2 + 9b^2)(4a^2 - 9b^2)$$
$$= (4a^2 + 9b^2)(2a + 3b)(2a - 3b).$$

Ex. 2. Resolve into factors $x^6 - y^6$.

$$x^6 - y^6 = (x^3 + y^3)(x^3 - y^3)$$
$$= (x+y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2).$$

Note. When an expression can be arranged either as the difference of two squares, or as the difference of two cubes, each of the methods explained in Arts. 98, 102 will be applicable. It will, however, be found simplest to first use the rule for resolving into factors the difference of two squares.

In all cases where an expression to be resolved contains a simple factor common to each of its terms, this should be first taken outside a bracket as explained in Art. 90.

Ex. 3. Resolve into factors $28x^4y + 64x^3y - 60x^2y$.

$$28x^4y + 64x^3y - 60x^2y = 4x^2y(7x^2 + 16x - 15)$$
$$= 4x^2y(7x - 5)(x + 3).$$

Ex. 4. Resolve into factors $x^3p^2 - 8y^3p^2 - 4x^3q^2 + 32y^3q^2$.

The expression $= p^2(x^3 - 8y^3) - 4q^2(x^3 - 8y^3)$
$$= (x^3 - 8y^3)(p^2 - 4q^2)$$
$$= (x - 2y)(x^2 + 2xy + 4y^2)(p + 2q)(p - 2q).$$

Ex. 5. Resolve into factors $4x^2 - 25y^2 + 2x + 5y$.

$$4x^2 - 25y^2 + 2x + 5y = (2x + 5y)(2x - 5y) + 2x + 5y$$
$$= (2x + 5y)(2x - 5y + 1).$$

EXAMPLES X. o.

Resolve into two or more factors:

1. $a^2 - y^2 - 2yz - z^2$.
2. $x^6 - y^6z^6$.
3. $6x^2 - x - 77$.
4. $729y^6 - 64x^6$.
5. $x^6 - 4096$.
6. $2mn + 2xy + m^2 + n^2 - x^2 - y^2$.
7. $33x^2 - 16x - 65$.
8. $a^4 + b^4 - c^4 - d^4 + 2a^2b^2 - 2c^2d^2$.
9. $m^3x + m^3y - n^3x - n^3y$.
10. $(a + b + c)^2 - (a - b - c)^2$.
11. $4 + 4x + 2ay + x^2 - a^2 - y^2$.
12. $x^2 - 10x - 119$.
13. $a^2 - b^2 - c^2 + d^2 - 2(ad - bc)$.
14. $x^2 - a^2 + y^2 - 2xy$.
15. $a^2 + x^2 - (y^2 + z^2) - 2(yz - ax)$.
16. $21x^2 + 82x - 39$.
17. $1 - a^2x^2 - b^2y^2 + 2abxy$.
18. $c^5d^3 - c^2 - a^2c^3d^3 + a^2$.
19. $a^2x^6 - a^2y^6 - b^2x^6 + b^2y^6$.
20. $x^2 - 6x - 247$.
21. $a^3x^2 - c^3x^2 - a^3y^2 + c^3y^2$.
22. $acx^2 - bcx + adx - bd$.
23. $a^2x - b^2x + a^2y - b^2y$.
24. $x^4 + 4x^2y^2z^2 + 4y^4z^4$.
25. $a^3b^3 + 512$.
26. $2x^2 + 17x + 35$.
27. $500x^2y - 20y^3$.
28. $a^5 - 8a^2b^3$.
29. $a^2x^3 - 16x^3y^2$.
30. $b^2 + c^2 - a^2 - 2bc$.
31. $5x^4 - 15x^3 - 90x^2$.
32. $14a^2x^3 - 35a^3x^2 + 14a^4x$.
33. $x^8 - 1$.
34. $1 - (m^2 + n^2) + 2mn$.
35. $75x^4 - 48a^4$.
36. $5a^4b^4 - 5ab$.
37. $8x^2y + 52xy + 60y$.
38. $3x^2y^2 + 26axy + 35a^2$.
39. $729a^7b - ab^7$.
40. $a^8x^6 - 64a^2y^6$.
41. $a^{12} - b^{12}$.
42. $24x^2y^2 - 30xy^3 - 36y^4$.
43. $(a + b)^4 - 1$.
44. $a^4 - (b + c)^4$.
45. $(c + d)^3 - 1$.

46. $1-(x-y)^3$.

47. $250(a-b)^3+2$.

48. $(c+d)^3+(c-d)^3$.

49. $8(x+y)^3-(2x-y)^3$.

50. x^2-4y^2+x-2y.

51. a^2-b^2+a-b.

52. $(a+b)^2+a+b$.

53. a^3+b^3+a+b.

54. a^2-9b^2+a+3b.

55. $4(x-y)^3-(x-y)$.

56. $x^4y-x^2y^3-x^3y^2+xy^4$.

57. $4a^2-9b^2+2a-3b$.

58. Resolve $x^{16}-y^{16}$ into five factors.

CONVERSE USE OF FACTORS.

104. The actual processes of multiplication and division can often be partially or wholly avoided by a skilful use of factors.

It should be observed that the formulæ which the student has seen exemplified in the preceding pages are just as useful in their converse as in their direct application. Thus the formula for resolving into factors the difference of two squares is equally useful as enabling us to write at once the product of the sum and the difference of two quantities.

Ex. 1. Multiply $2a+3b-c$ by $2a-3b+c$.

These expressions may be arranged thus:

$$2a+(3b-c) \text{ and } 2a-(3b-c).$$

Hence the product $=\{2a+(3b-c)\}\{2a-(3b-c)\}$

$$=(2a)^2-(3b-c)^2 \qquad \text{[Art. 98.]}$$

$$=4a^2-(9b^2-6bc+c^2)$$

$$=4a^2-9b^2+6bc-c^2.$$

Ex. 2. Divide the product of $2x^2+x-6$, and $6x^2-5x+1$ by $3x^2+5x-2$.

Denoting the division by writing the divisor under the dividend (Art. 53), with a horizontal line between them, the required quotient

$$=\frac{(2x^2+x-6)(6x^2-5x+1)}{3x^2+5x-2}$$

$$=\frac{(2x-3)(x+2)(3x-1)(2x-1)}{(3x-1)(x+2)}$$

$$=(2x-3)(2x-1).$$

Ex. 3. Prove the identity

$$17(5x+3a)^2-2(40x+27a)(5x+3a)=25x^2-9a^2.$$

Since each term of the first expression contains the factor $5x+3a$, the first side

$$\begin{aligned}&=(5x+3a)\{17(5x+3a)-2(40x+27a)\}\\&=(5x+3a)(85x+51a-80x-54a)\\&=(5x+3a)(5x-3a)\\&=25x^2-9a^2.\end{aligned}$$

Ex. 4. Show that $(2x+3y-z)^3+(3x+7y+z)^3$ is divisible by $5(x+2y)$.

The given expression is of the form A^3+B^3, and therefore has a divisor of the form $A+B$.

Therefore $(2x+3y-z)^3+(3x+7y+z)^3$

is divisible by $(2x+3y-z)+(3x+7y+z)$,

that is, by $5x+10y$,

or by $5(x+2y)$.

EXAMPLES X. p.

Find the product of

1. $2x-7y+3z$ and $2x+7y-3z$.
2. $3x^2-4xy+7y^2$ and $3x^2+4xy+7y^2$.
3. $5x^2+5xy-9y^2$ and $5x^2-5xy-9y^2$.
4. $7x^2-8xy+3y^2$ and $7x^2+8xy-3y^2$.
5. $x^3+2x^2y+2xy^2+y^3$ and $x^3-2x^2y+2xy^2-y^3$.
6. $(x+y)^2+2(x+y)+4$ and $(x+y)^2-2(x+y)+4$.
7. Multiply the square of $a+3b$ by $a^2-6ab+9b^2$.
8. Divide $(4x+3y-2z)^2-(3x-2y+3z)^2$ by $x+5y-5z$.
9. Divide $x^8+16a^4x^4+256a^8$ by $x^2+2ax+4a^2$.
10. Divide $(x^2+7x+10)(x+3)$ by x^2+5x+6.
11. Divide $(3x+4y-2z)^2-(2x+3y-4z)^2$ by $x+y+2z$.

Prove the following identities:

12. $(a+b)^3-(a-b)^2(a+b)=4ab(a+b)$.
13. $c^4-d^4-(c-d)^3(c+d)=2cd(c^2-d^2)$.
14. $(x+y)^4-3xy(x+y)^2=(x+y)(x^3+y^3)$.

15. Show that the square of $x + 1$ exactly divides

$$(x^3 + x^2 + 4)^3 - (x^3 - 2x + 3)^3.$$

16. Show that $(3x^2 - 7x + 2)^3 - (x^2 - 8x + 8)^3$ is divisible by $2x - 3$, and by $x + 2$.

105. The Factor Theorem. *If any rational and integral expression containing x becomes equal to 0 when a is written for x, it is exactly divisible by $x - a$.*

Let P stand for the expression. Divide P by $x - a$ until the remainder no longer contains x. Let R denote this remainder, and Q the quotient obtained. Then

$$P = Q(x - a) + R.$$

Since this equation is true for all values of x, we will assume that x equals a. By hypothesis, the substitution of a for x makes P equal to 0; thus,

$$0 = Q(0) + R.$$
$$\therefore\ R = 0.$$

As the remainder is 0, the expression is exactly divisible by $x - a$.

The following examples illustrate the application of this principle:

Ex. 1. Resolve into factors $x^3 + 3x^2 - 13x - 15$.

By trial we find that this expression becomes 0 when $x = 3$; hence, $x - 3$ is a factor. Dividing by $x - 3$, we obtain the quotient

$$x^2 + 6x + 5.$$

The factors of this expression are easily seen to be $x + 1$ and $x + 5$; hence,

$$x^3 + 3x^2 - 13x - 15 = (x - 3)(x + 1)(x + 5).$$

Ex. 2. Resolve into factors $x^3 + 6x^2 + 11x + 6$.

It is evident that substituting a positive number for x will not make the expression equal to 0. By substituting -1, however, for x, the expression becomes $-1 + 6 - 11 + 6$, or 0; hence,

$$x^3 + 6x^2 + 11x + 6$$

is divisible by $x + 1$. Dividing by $x + 1$, we obtain the quotient $x^2 + 5x + 6$, and factoring this expression we have

$$x^3 + 6x^2 + 11x + 6 = (x + 1)(x + 2)(x + 3).$$

NOTE. The student should notice that the only numerical values that need be substituted for x are the *factors* of the *last term* of the expression, and that we *change the sign* of the factor substituted before connecting it with x. Thus, in Ex. 1, the factor 3 gives a divisor $x-3$, and in Ex. 2, the factor -1 gives a divisor $x+1$.

Ex. **3.** Without actual division, show that $5x^5-6x^3+1$ is divisible by $x-1$.

If the expression is divisible by $x-1$, it will become 0, or "vanish," when 1 is substituted for x. Making this substitution, we obtain 0; hence the division is possible.

EXAMPLES X. r.

Without actual division, show that $x-2$ is a factor of each of the following expressions:

1. x^3-5x+2. **3.** $x^3-7x^2+16x-12$.

2. x^3+x^2-4x-4. **4.** $x^3-8x^2+17x-10$.

Determine by inspection whether $x+3$ is a factor of any of the following expressions:

5. x^3-7x+6. **7.** x^3+6x+6.

6. $x^3+6x^2+11x+6$. **8.** x^3+3x^2+x+3.

9. Show that $32x^{10}-33x^5+1$ is divisible by $x-1$.

Resolve into factors.

10. $2x^3+4x^2-2x-4$. **11.** $3x^3-6x^2-3x+6$.

106. We shall employ the **Factor Theorem** in giving general proofs of the statements made in Art. 62.

We suppose n to be a positive integer.

(I.) x^n-y^n *is* **always** *divisible by* $x-y$.

By Art. 105, x^n-y^n is exactly divisible by $x-y$ if the substitution of y for x in the expression x^n-y^n gives zero as a result. Making this substitution we have

$$x^n-y^n=y^n-y^n=0.$$

Therefore x^n-y^n is *always* divisible by $x-y$.

(II.) x^n-y^n *is divisible by* $x+y$ *when* n **is even.**

If this be true, the substitution of $-y$ for x in the expression x^n-y^n gives zero as a result. Making this substitution we have

$$x^n-y^n=(-y)^n-y^n.$$

When n *is even*, this expression becomes $y^n - y^n$, or zero. Therefore $x^n - y^n$ is divisible by $x + y$ when n *is even*.

(III.) $x^n + y^n$ *is* **never** *divisible by* $x - y$.

Here the substitution of y for x in the expression $x^n + y^n$ gives

$$x^n + y^n = y^n + y^n = 2\,y^n.$$

As this expression is not zero, $x^n + y^n$ is *never* divisible by $x - y$.

(IV.) $x^n + y^n$ is divisible by $x + y$ when ***n*** **is odd.**

Here the substitution of $-y$ for x in the expression $x^n + y^n$ gives

$$x^n + y^n = (-y)^n + y^n.$$

When n *is odd*, this expression becomes $-y^n + y^n$, or zero. Therefore $x^n + y^n$ is divisible by $x + y$ when n *is odd*.

The results of the present article may be conveniently stated as follows:

(i.) For all positive integral values of n,

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}.$$

(ii.) When n is odd,

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots + y^{n-1}).$$

(iii.) When n is even,

$$x^n - y^n = (x + y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \cdots - y^{n-1}).$$

107. We shall now discuss some cases of greater difficulty, and also show how certain expressions of frequent occurrence, which are not integral, may be separated into factors. The student may omit this portion of the chapter until reading the subject a second time.

Ex. 1. Resolve $a^9 - 64\,a^3 - a^6 + 64$ into six factors.

The expression

$= a^3(a^6 - 64) - (a^6 - 64)$

$= (a^6 - 64)(a^3 - 1)$

$= (a^3 + 8)(a^3 - 8)(a^3 - 1)$

$= (a + 2)(a^2 - 2\,a + 4)(a - 2)(a^2 + 2\,a + 4)(a - 1)(a^2 + a + 1).$

Ex. 2. $$a(a-1)x^2-(a-b-1)xy-b(b+1)y^2$$
$$=\{ax-(b+1)y\}\{(a-1)x+by\}.$$

108. From Ex. 2, Art. 59, we see that the quotient of $a^3+b^3+c^3-3abc$ by $a+b+c$ is $a^2+b^2+c^2-bc-ca-ab$.
Thus $$a^3+b^3+c^3-3abc$$
$$=(a+b+c)(a^2+b^2+c^2-bc-ca-ab) \quad . \quad . \quad (1).$$

This result is important and should be carefully remembered. We may note that the expression on the left consists of the sum of the cubes of three quantities a, b, c, diminished by three times the product abc. Whenever an expression admits of a similar arrangement, the above formula will enable us to resolve it into factors.

Ex. 1. Resolve into factors $a^3-b^3+c^3+3abc$.
$$a^3-b^3+c^3+3abc=a^3+(-b)^3+c^3-3a(-b)c$$
$$=(a-b+c)(a^2+b^2+c^2+bc-ca+ab),$$
$-b$ taking the place of b in formula (1).

Ex. 2.
$$x^3-8y^3-27-18xy=x^3+(-2y)^3+(-3)^3-3x(-2y)(-3)$$
$$=(x-2y-3)(x^2+4y^2+9-6y+3x+2xy).$$

109. Expressions which can be put into the form $x^3 \pm \dfrac{1}{y^3}$ may be separated into factors by the rules for resolving the sum or the difference of two cubes. [Art. 102.]

Ex. 1. $$\frac{8}{a^3}-27b^6=\left(\frac{2}{a}\right)^3-(3b^2)^3$$
$$=\left(\frac{2}{a}-3b^2\right)\left(\frac{4}{a^2}+\frac{6b^2}{a}+9b^4\right).$$

Ex. 2. Resolve $a^2x^3-\dfrac{8a^2}{y^3}-x^3+\dfrac{8}{y^3}$ into four factors.
$$a^2x^3-\frac{8a^2}{y^3}-x^3+\frac{8}{y^3}=x^3(a^2-1)-\frac{8}{y^3}(a^2-1)$$
$$=(a^2-1)\left(x^3-\frac{8}{y^3}\right)$$
$$=(a+1)(a-1)\left(x-\frac{2}{y}\right)\left(x^2+\frac{2x}{y}+\frac{4}{y^2}\right).$$

EXAMPLES X. s.

Resolve into two or more factors:

1. $x^2y + 3xy^2 - 3x^3 - y^3$.
2. $4mn^2 - 20n^3 + 45nm^2 - 9m^3$.
3. $ab(x^2+1) + x(a^2+b^2)$.
4. $y^2z^2(x^4-1) + x^2(y^4-z^4)$.
5. $a^3 + (a+b)ax + bx^2$.
6. $pn(m^2+1) - m(p^2+n^2)$.
7. $6bx(a^2+1) - a(4x^2+9b^2)$.
8. $(2a^2+3y^2)x + (2x^2+3a^2)y$.
9. $(2x^2-3a^2)y + (2a^2-3y^2)x$.
10. $a(a-1)x^2 + (2a^2-1)x + a(a+1)$.
11. $3x^2 - (4a+2b)x + a^2 + 2ab$.
12. $2a^2x^2 - 2(3b-4c)(b-c)y^2 + abxy$.
13. $(a^2-3a+2)x^2 + (2a^2-4a+1)x + a(a-1)$.
14. $a(a+1)x^2 + (a+b)xy - b(b-1)y^2$.
15. $b^3 + c^3 - 1 + 3bc$.
16. $a^3 + 8c^3 + 1 - 6ac$.
17. $a^3 + b^3 + 8c^3 - 6abc$.
18. $a^3 - 27b^3 + c^3 + 9abc$.
19. $a^3 - b^3 - c^3 - 3abc$.
20. $8a^3 + 27b^3 + c^3 - 18abc$.
21. $\dfrac{27}{a^3b^3} - 1$.
22. $216a^3 - \dfrac{b^3}{8}$.
23. $\dfrac{x^3}{125} + y^3$.
24. $\dfrac{m^3n^3}{729} - 1$.
25. $\dfrac{a^3b^3}{125} + 1000$.
26. $\dfrac{x^3}{512} - \dfrac{64}{x^3}$.
27. Resolve $x^8 + 81x^4 + 6561$ into three factors.
28. Resolve $(a^4 - 2a^2b^2 - b^4)^2 - 4a^4b^4$ into four factors.
29. Resolve $4(ab+cd)^2 - (a^2+b^2-c^2-d^2)^2$ into four factors.
30. Resolve $x^8 - \dfrac{1}{256}$ into four factors.
31. Resolve $x^{18} - y^{18}$ into six factors.

Resolve into four factors:

32. $\dfrac{a^3}{x^2} - 8x - a^3 + 8x^3$.
33. $x^9 + x^3y^6 - 8x^6y^3 - 8y^9$.
34. $x^9 + x^6 + 64x^3 + 64$.
35. $4a - 9b + \dfrac{4b^3}{a^2} - \dfrac{9a^3}{b^2}$.
36. $\dfrac{xy^3}{72} - \dfrac{x^3y^5}{32} - \dfrac{1}{9x^2} + \dfrac{y^2}{4}$.
37. $x^6 - 25x^2 + 6\frac{1}{4} - \dfrac{1}{4}x^4$.

Resolve into five factors:

38. $x^7 + x^4 - 16x^3 - 16$.
39. $16x^7 - 81x^3 - 16x^4 + 81$.

CHAPTER XI.

Highest Common Factor.

110. Definition. The **Highest Common Factor** of two or more algebraic expressions is the expression of *highest dimensions* (Art. 29) which divides each of them without remainder.

The abbreviation H. C. F. is sometimes used instead of the words *highest common factor.*

SIMPLE EXPRESSIONS.

111. The H. C. F. can be written by inspection.

Ex. 1. The highest common factor of a^4, a^3, a^2, a^6 is a^2.

Ex. 2. The highest common factor of a^3b^4, ab^5c^2, a^2b^7c is ab^4; for a is the highest power of a that will divide a^3, a, a^2; b^4 is the highest power of b that will divide b^4, b^5, b^7; and c is not a *common* factor.

112. If the expressions have numerical coefficients, find by Arithmetic their greatest common measure, and prefix it as a coefficient to the algebraic highest common factor.

Ex. The highest common factor of $21\,a^4x^3y$, $35\,a^2x^4y$, $28\,a^3xy^4$ is $7\,a^2xy$; for it consists of the product of

(1) the numerical greatest common measure of the coefficients;

(2) the highest power of each letter which divides every one of the given expressions.

EXAMPLES XI. a.

Find the highest common factor of

1. $4\,ab^2$, $2\,a^2b$.

2. $3\,x^2y^2$, x^3y^2.

3. $6\,xy^2z$, $8\,x^2y^3z^2$.

4. abc, $2\,ab^2c$.

5. $5\,a^3b^3$, $15\,abc^2$.

6. $9\,x^2y^2z^2$, $12\,xy^3z$.

7. $4a^2b^3c^2$, $6a^3b^2c^3$.

8. $7a^2b^4c^5$, $14ab^2c^3$.

9. $49ax^2$, $63ay^2$, $56az^2$.

10. $17ab^2c$, $34a^2bc$, $51abc^2$.

11. $a^3x^2y^2$, b^3xy^2, c^3x^2y.

12. $25xy^2z$, $100x^2yz$, $125xy$.

13. a^2bpxy, b^2qxy, a^3bxr^2.

14. $15a^5b^3c^7$, $60a^3b^7c^6$, $25a^4b^5c^2$.

15. $35a^2c^3b$, $42a^3cb^2$, $30ac^2b^3$.

16. $24a^3b^2c^3$, $16a^3b^4c^2$, $40a^2b^3c^5$.

COMPOUND EXPRESSIONS.

113. H. C. F. of Compound Expressions which can be factored by Inspection. The method employed is similar to that of the preceding article.

Ex. 1. Find the highest common factor of

$$4cx^3 \text{ and } 2cx^3 + 4c^2x^2.$$

It will be easy to pick out the common factors if the expressions are arranged as follows:

$$4cx^3 = 4cx^3,$$
$$2cx^3 + 4c^2x^2 = 2cx^2(x + 2c);$$

therefore the H. C. F. is $2cx^2$.

Ex. 2. Find the highest common factor of

$$3a^2 + 9ab,\ a^3 - 9ab^2,\ a^3 + 6a^2b + 9ab^2.$$

Resolving each expression into its factors, we have

$$3a^2 + 9ab = 3a(a + 3b),$$
$$a^3 - 9ab^2 = a(a + 3b)(a - 3b),$$
$$a^3 + 6a^2b + 9ab^2 = a(a + 3b)(a + 3b);$$

therefore the H. C. F. is $a(a + 3b)$.

114. When there are two or more expressions containing different powers of the same *compound* factor, the student should be careful to notice that the highest common factor must contain the highest power of the compound factor which is common to all the given expressions.

Ex. 1. The highest common factor of

$$x(a - x)^2,\ a(a - x)^3, \text{ and } 2ax(a - x)^5 \text{ is } (a - x)^2.$$

Ex. 2. Find the highest common factor of

$$ax^2 + 2a^2x + a^3,\ 2ax^2 - 4a^2x - 6a^3,\ 3(ax + a^2)^2.$$

Resolving the expressions into factors, we have

$$ax^2 + 2a^2x + a^3 = a(x^2 + 2ax + a^2)$$
$$= a(x + a)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$2ax^2 - 4a^2x - 6a^3 = 2a(x^2 - 2ax - 3a^2)$$
$$= 2a(x + a)(x - 3a) \quad . \quad . \quad . \quad (2),$$
$$3(ax + a^2)^2 = 3a^2(x + a)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Therefore from (1), (2), (3), by inspection, the highest common factor is $a(x + a)$.

EXAMPLES XI. b.

Find the highest common factor of

1. $a^2 + ab,\ a^2 - b^2.$

2. $(x + y)^2,\ x^2 - y^2.$

3. $2x^2 - 2xy,\ x^3 - x^2y.$

4. $6x^2 - 9xy,\ 4x^2 - 9y^2.$

5. $x^3 + x^2y,\ x^3 + y^3.$

6. $a^3b - ab^3,\ a^5b^2 - a^2b^5.$

7. $a^3 - a^2x,\ a^3 - ax^2,\ a^4 - ax^3.$

8. $a^2 - 4x^2,\ a^2 + 2ax.$

9. $a^2 - x^2,\ a^2 - ax,\ a^2x - ax^2.$

10. $4x^2 + 2xy,\ 12x^2y - 3y^3.$

11. $20x - 4,\ 50x^2 - 2.$

12. $6bx + 4by,\ 9cx + 6cy.$

13. $x^2 + x,\ (x + 1)^2,\ x^3 + 1.$

14. $xy - y,\ x^4y - xy.$

15. $x^2 - 2xy + y^2,\ (x - y)^3.$

16. $x^3 + a^2x,\ x^4 - a^4.$

17. $x^3 + 8y^3,\ x^2 + xy - 2y^2.$

18. $x^4 - 27a^3x,\ (x - 3a)^2.$

19. $x^2 + 3x + 2,\ x^2 - 4.$

20. $x^2 - x - 20,\ x^2 - 9x + 20.$

21. $x^2 - 18x + 45,\ x^2 - 9.$

22. $2x^2 - 7x + 3,\ 3x^2 - 7x - 6.$

23. $x^5 - xy^2,\ x^3 + x^2y + xy + y^2.$

24. $a^3x - a^2bx - 6ab^2x,\ a^2bx^2 - 4ab^2x^2 + 3b^3x^2.$

25. $2x^2 + 9x + 4,\ 2x^2 + 11x + 5,\ 2x^2 - 3x - 2.$

26. $3x^4 + 8x^3 + 4x^2,\ 3x^5 + 11x^4 + 6x^3,\ 3x^4 - 16x^3 - 12x^2.$

27. $2x^4 + 5x^3 + 3x^2,\ 6x^4 + 13x^3 + 6x^2,\ 2x^4 - 7x^3 - 15x^2.$

28. $12x + 6x^2 + 6,\ 6x + 3x^2 + 3,\ 18x + 3x^2 + 15.$

29. $x^4 + 4x^2 + 3,\ x^4 + 5x^2 + 6,\ 3x^4 + 11x^2 + 6.$

30. $2a^2 + 7ad + 6d^2,\ 2a^2 + 9ad + 9d^2,\ 6a^2 + 11ad + 3d^2.$

31. $2x^2 + 8xy + 6y^2,\ 4x^2 + 14xy + 6y^2,\ 2x^2 + 10xy + 12y^2.$

115. H. C. F. of Compound Expressions which cannot be factored by Inspection. To find the highest common factor in such cases, we adopt a method analogous to that used in Arithmetic for finding the greatest common measure of two or more numbers.

NOTE. The term *greatest common measure* is sometimes used instead of *highest common factor*; but, strictly speaking, the term *greatest common measure* ought to be confined to arithmetical quantities; for the highest common factor is not necessarily the greatest common measure in all cases, as will appear later. (Art. 121.)

116. We begin by working out examples illustrative of the algebraic process of finding the highest common factor, postponing for the present the complete proof of the rules we use. But we may conveniently enunciate two principles, which the student should bear in mind in reading the examples which follow.

I. *If an expression contains a certain factor, any multiple of the expression is divisible by that factor.*

II. *If two expressions have a common factor, it will divide their sum and their difference; and also the sum and the difference of any multiples of them.*

Ex. Find the highest common factor of

$$4x^3-3x^2-24x-9 \text{ and } 8x^3-2x^2-53x-39.$$

$$\begin{array}{r|r|r|l}
x & 4x^3-3x^2-24x-9 & 8x^3-2x^2-53x-39 & 2\\
 & \underline{4x^3-5x^2-21x} & \underline{8x^3-6x^2-48x-18} & \\
2x & 2x^2-3x-9 & 4x^2-5x-21 & 2\\
 & \underline{2x^2-6x} & \underline{4x^2-6x-18} & \\
3 & 3x-9 & x-3 & \\
 & \underline{3x-9} & &
\end{array}$$

Therefore the H. C. F. is $x-3$.

EXPLANATION. First arrange the given expressions according to descending or ascending powers of x. The expressions so arranged having their first terms of the same order, we take for divisor that whose highest power has the smaller coefficient. Arrange the work in parallel columns as above. When the first remainder $4x^2-5x-21$ is made the divisor we put the quotient x to the *left* of the dividend. Again, when the second remainder $2x^2-3x-9$ is in turn made the divisor, the quotient 2 is placed to the *right*; and so on. As in Arithmetic, the last divisor $x-3$ is the highest common factor required.

117. This method is only useful to determine the *compound* factor of the highest common factor. *Simple* factors of the given expressions must be first removed from them, and the highest common factor of these, if any, must be observed and multiplied into the *compound* factor given by the rule.

Ex. Find the highest common factor of

$$24x^4 - 2x^3 - 60x^2 - 32x \text{ and } 18x^4 - 6x^3 - 39x^2 - 18x.$$

We have $24x^4 - 2x^3 - 60x^2 - 32x = 2x(12x^3 - x^2 - 30x - 16)$,

and $\qquad 18x^4 - 6x^3 - 39x^2 - 18x = 3x(6x^3 - 2x^2 - 13x - 6)$.

Also $2x$ and $3x$ have the common factor x. Removing the simple factors $2x$ and $3x$, and *reserving* their common factor x, we continue as in Art. 116.

$$
\begin{array}{r|r|r|l}
2x & 6x^3-2x^2-13x-6 & 12x^3-x^2-30x-16 & 2 \\
 & 6x^3-8x^2-8x & 12x^3-4x^2-26x-12 & \\
\cline{2-3}
2 & 6x^2-5x-6 & 3x^2-4x-4 & x \\
 & 6x^2-8x-8 & 3x^2+2x & \\
\cline{2-3}
 & 3x+2 & -6x-4 & -2 \\
 & & -6x-4 & \\
\cline{3-3}
\end{array}
$$

Therefore the H. C. F. is $x(3x+2)$.

118. So far the process of Arithmetic has been found exactly applicable to the algebraic expressions we have considered. But in many cases certain modifications of the arithmetical method will be found necessary. These will be more clearly understood if it is remembered that, at every stage of the work, the remainder must contain as a factor of itself the highest common factor we are seeking. [See Art. 116, I & II.]

Ex. **1.** Find the highest common factor of

$$3x^3 - 13x^2 + 23x - 21 \text{ and } 6x^3 + x^2 - 44x + 21.$$

$$
\begin{array}{r|r|l}
3x^3-13x^2+23x-21 & 6x^3+x^2-44x+21 & 2 \\
 & 6x^3-26x^2+46x-42 & \\
\cline{2-2}
 & 27x^2-90x+63 & \\
\end{array}
$$

Here on making $27x^2 - 90x + 63$ a divisor, we find that it is not contained in $3x^3 - 13x^2 + 23x - 21$ with an *integral* quotient. But noticing that $27x^2-90x+63$ may be written in the form $9(3x^2-10x+7)$, and also bearing in mind that every remainder in the course of the work contains the H. C. F., we conclude that the H. C. F. we are seeking is contained in $9(3x^2 - 10x + 7)$. But the two original expressions have no *simple* factors, therefore their H. C. F. can have none. We may therefore *reject* the factor 9 and go on with divisor $3x^2 - 10x + 7$. Resuming the work, we have

$$\begin{array}{r|r|r|l}
x & 3x^3 - 13x^2 + 23x - 21 & 3x^2 - 10x + 7 & x \\
 & 3x^3 - 10x^2 + 7x \phantom{{}-21} & 3x^2 - 7x \phantom{{}+7} & \\
\cline{2-3}
-1 & -3x^2 + 16x - 21 & -3x + 7 & -1 \\
 & -3x^2 + 10x - 7 & -3x + 7 & \\
\cline{2-3}
 & 2)6x - 14 & & \\
 & 3x - 7 & &
\end{array}$$

Therefore the highest common factor is $3x - 7$.

The factor 2 has been removed on the same grounds as the factor 9 above.

Ex. **2**. Find the highest common factor of

$$2x^3 + x^2 - x - 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and

$$3x^3 - 2x^2 + x - 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

As the expressions stand we cannot begin to divide one by the other without using a fractional quotient. The difficulty may be obviated by *introducing* a suitable factor, just as in the last case we found it useful to remove a factor when we could no longer proceed with the division in the ordinary way. The given expressions have no common *simple* factor, hence their H. C. F. cannot be affected if we multiply either of them by any simple factor.

Multiply (2) by 2, and use (1) as a divisor:

$$\begin{array}{r|r|r|l}
 & 2x^3 + x^2 - x - 2 & 6x^3 - 4x^2 + 2x - 4 & 3 \\
 & 7 \phantom{{}+ x^2 - x - 2} & 6x^3 + 3x^2 - 3x - 6 & \\
\cline{2-3}
-2x & 14x^3 + 7x^2 - 7x - 14 & -7x^2 + 5x + 2 & \\
 & 14x^3 - 10x^2 - 4x \phantom{{}-14} & 17 & \\
\cline{2-3}
17x & 17x^2 - 3x - 14 & -119x^2 + 85x + 34 & -7 \\
 & 17x^2 - 17x \phantom{{}-14} & -119x^2 + 21x + 98 & \\
\cline{2-3}
14 & 14x - 14 & 64)64x - 64 & \\
 & 14x - 14 & x - 1 &
\end{array}$$

Therefore the H.C.F. is $x - 1$.

After the first division the factor 7 is introduced because the first remainder $-7x^2 + 5x + 2$ will not divide $2x^3 + x^2 - x - 2$.

At the next stage the factor 17 is introduced for a similar reason, and finally the factor 64 is removed as explained in Ex. 1.

From these examples it appears that we may multiply or divide either of the given expressions, or any of the remainders which occur in the course of the work, by any factor which does not divide both of the given expressions.

NOTE. If, in Ex. 2, the expressions had been arranged in *ascending* powers of x, it would have been found unnecessary to introduce a numerical factor in the course of the work.

119. The use of the **Factor Theorem** (Art. 105) often lessens, in a very marked degree, the work of finding the highest common factor. Thus in Ex. 2 of the preceding article it is easily seen that both expressions become equal to 0 when 1 is substituted for x, hence $x-1$ is a factor. Dividing the first of the given expressions by $x-1$, we obtain a quotient $2x^2+3x+2$. It is evident that this will not divide the second expression, hence $x-1$ is the H. C. F.

120. When the **Method of Division by Detached Coefficients** (Art. 63) is employed in finding the H. C. F., the following is a convenient arrangement.

Ex. Find the H. C. F. of

$$x^4+3x^3+12x-16 \text{ and } x^3-13x+12.$$

We write the literal factors of the dividend until we reach a term of the same degree as the first term of the divisor.

$$\begin{array}{r|rrrrr}
x^3 & x^4 & +3x^3 & +\ 0 & +12 & -16 \\
0 & & 0 & +13 & -12 & \\
+13 & & & 0 & +39 & -36 \\
-12 & & & & & \\
\hline
 & x & +3; & 13 & +39 & -52
\end{array}$$

The addition of the terms in the third column gives $13x^2$, which is of lower *degree* than the first term of the divisor, hence we can proceed no further with the division and have for a remainder $13x^2+39x-52$. Removing from this remainder the factor 13, as it is not a factor of the given expressions, we have for a second divisor x^2+3x-4. The first divisor, as written *before the signs were changed*, forms the second dividend:

$$\begin{array}{r|rrrr}
x^2 & x^3 & +0x^2 & -13 & +12 \\
-3 & & -3 & +\ 4 & \\
+4 & & & +\ 9 & -12 \\
\hline
 & x & -3; & 0 & 0
\end{array}$$

since there is no remainder, the last divisor, as written *before the signs were changed*, is the H. C. F. Thus x^2+3x-4 is the H. C. F.

121. Let the two expressions in Ex. 2, Art. 118, be written in the form

$$2x^2 + x^2 - x - 2 = (x-1)(2x^2 + 3x + 2),$$
$$3x^3 - 2x^2 + x - 2 = (x-1)(3x^2 + x + 2).$$

Then their highest common factor is $x - 1$, and therefore $2x^2 + 3x + 2$ and $3x^2 + x + 2$ *have no algebraic common divisor.* If, however, we put $x = 6$, then

$$2x^3 + x^2 - x - 2 = 460,$$

and $$3x^3 - 2x^2 + x - 2 = 580;$$

and the greatest common measure of 460 and 580 is 20; whereas 5 is the numerical value $x - 1$, the algebraic highest common factor. Thus the numerical values of the algebraic highest common factor and of the arithmetical greatest common measure do not in this case agree.

The reason may be explained as follows: when $x = 6$, the expressions $2x^2 + 3x + 2$ and $3x^2 + x + 2$ become equal to 92 and 116 respectively, and have a common arithmetical factor 4; whereas the expressions have no algebraic common factor.

It will thus often happen that the highest common factor of two expressions and their numerical greatest common measure, when the letters have particular values, are not the same; for this reason the term greatest common measure is inappropriate when applied to algebraic quantities.

EXAMPLES XI. c.

Find the highest common factor of the following expressions:

1. $x^3 + 2x^2 - 13x + 10,\ x^3 + x^2 - 10x + 8.$

2. $x^3 - 5x^2 - 99x + 40,\ x^3 - 6x^2 - 86x + 35.$

3. $x^3 + 2x^2 - 8x - 16,\ x^3 + 3x^2 - 8x - 24.$

4. $x^3 - x^2 - 5x - 3,\ x^3 - 4x^2 - 11x - 6.$

5. $x^3 + 3x^2 - 8x - 24,\ x^3 + 3x^2 - 3x - 9.$

6. $a^3 - 5a^2x + 7ax^2 - 3x^3,\ a^3 - 3ax^2 + 2x^3.$

7. $2x^3 - 5x^2 + 11x + 7,\ 4x^3 - 11x^2 + 25x + 7.$

8. $2x^3 + 4x^2 - 7x - 14,\ 6x^3 - 10x^2 - 21x + 35.$

9. $3x^4 - 3x^3 - 2x^2 - x - 1,\ 9x^4 - 3x^3 - x - 1.$

10. $3x^3 - 3ax^2 + 2a^2x - 2a^3,\ 3x^3 + 12ax^2 + 2a^2x + 8a^3.$

11. $2x^3 - 9ax^2 + 9a^2x - 7a^3,\ 4x^3 - 20ax^2 + 20a^2x - 16a^3.$

12. $10x^3 + 25ax^2 - 5a^3,\ 4x^3 + 9ax^2 - 2a^2x - a^3.$

13. $24x^4y + 72x^3y^2 - 6x^2y^3 - 90xy^4,\ 6x^4y^2 + 13x^3y^3 - 4x^2y^4 - 15xy^5.$

14. $4x^5a^2 + 10x^4a^3 - 60x^3a^4 + 54x^2a^5,\ 24x^5a^3 + 30x^3a^5 - 126x^2a^6.$

15. $4x^5 + 14x^4 + 20x^3 + 70x^2,\ 8x^7 + 28x^6 - 8x^5 - 12x^4 + 56x^3.$

16. $72x^3 - 12ax^2 + 72a^2x - 420a^3,\ 18x^3 + 42ax^2 - 282a^2x + 270a^3.$

17. $x^5 - x^3 - x + 1,\ x^7 + x^6 + x^4 - 1.$

18. $1 + x + x^3 - x^5,\ 1 - x^4 - x^6 + x^7.$

122. The statements of Art. 116 may be proved as follows:

I. If F divides A it will also divide mA.

For suppose $A = aF$, then $mA = maF$.

Thus F is a factor of mA.

II. If F divides A and B, then it will divide $mA \pm nB$.

For suppose $A = aF$, $B = bF$,

then
$$\begin{aligned} mA \pm nB &= maF \pm nbF \\ &= F(ma \pm nb). \end{aligned}$$

Thus F divides $mA \pm nB$.

123. We may now enunciate and prove the rule for finding the highest common factor of any two compound algebraic expressions.

We suppose that any simple factors are first removed. (See Example, Art. 117.)

Let A and B be the two expressions after the simple factors have been removed. Let them be arranged in descending or ascending powers of some common letter; also let the highest power of that letter in B be not less than the highest power in A.

Divide B by A; let p be the quotient, and C the remainder. Suppose C to have a *simple* factor m. Remove this factor, and so obtain a new divisor D. Further, suppose that in order to make A divisible by D it is necessary to multiply A by a *simple* factor n. Let q be the next quotient and E the remainder. Finally, divide D by E; let r be the quotient, and suppose that there is no remainder. Then E will be the H. C. F. required.

The work will stand thus:

$$\begin{array}{llll}
A)B(p & & & \\
\;\;\underline{pA} & & & \\
m)\underline{C\;\;} & & & \\
 & D)nA(q & & \\
 & \;\;\;\underline{qD} & & \\
 & & E)D(r & \\
 & & \;\;\;\underline{rE} &
\end{array}$$

First, to show that E is *a* common factor of A and B.

By examining the steps of the work, it is clear that E divides D, therefore also qD; therefore $qD+E$, therefore nA; therefore A, since n is a *simple* factor.

Again E divides D, therefore mD, that is, C. And since E divides A and C, it also divides $pA+C$, that is, B. Hence E divides both A and B.

Secondly, to show that E is the *highest* common factor.

If not, let there be a factor X of higher dimensions than E.

Then X divides A and B, therefore $B-pA$, that is, C; therefore D (since m is a *simple* factor); therefore $nA-qD$, that is, E.

Thus X divides E; which is impossible, since by hypothesis, X is of higher dimensions than E.

Therefore E is the highest common factor.

124. The highest common factor of three expressions A, B, C may be obtained as follows:

First determine F the highest common factor of A and B; next find G the highest common factor of F and C; then G will be the required highest common factor of A, B, C.

For F contains *every* factor which is common to A and B, and G is the highest common factor of F and C. Therefore G is the highest common factor of A, B, C.

CHAPTER XII.

Lowest Common Multiple.

125. Definition. The **Lowest Common Multiple** of two or more algebraic expressions is the expression of *lowest dimensions* which is divisible by each of them without remainder.

The abbreviation L. C. M. is sometimes used instead of the words *lowest common multiple.*

SIMPLE EXPRESSIONS.

126. The L. C. M. can be written by inspection.

Ex. 1. The lowest common multiple of a^4, a^3, a^2, a^6 is a^6.

Ex. 2. The lowest common multiple of a^3b^4, ab^5, a^2b^7 is a^3b^7; for a^3 is the lowest power of a that is divisible by each of the quantities a^3, a, a^2; and b^7 is the lowest power of b that is divisible by each of the quantities b^4, b^5, b^7.

127. If the expressions have numerical coefficients, find by Arithmetic their least common multiple, and prefix it as a coefficient to the algebraic lowest common multiple.

Ex. The lowest common multiple of $21\,a^4x^3y$, $35\,a^2x^4y$, $28\,a^3xy^4$ is $420\,a^4x^4y^4$; for it consists of the product of

(1) the numerical least common multiple of the coefficients;

(2) the lowest power of each letter which is divisible by every power of that letter occurring in the given expressions.

EXAMPLES XII. a.

Find the lowest common multiple of

1. x^3y^2, xyz.
2. $3\,x^2yz$, $4\,x^3y^3$.
3. $5\,a^2bc^3$, $4\,ab^2c$.
4. $12\,ab$, $8\,xy$.
5. ac, bc, ab.
6. a^2c, bc^2, cb^2.
7. $2\,x$, $3\,y$, $4\,z$.
8. $3\,x^2$, $4\,y^2$, $3\,z^2$.
9. $7\,a^2$, $2\,ab$, $3\,b^3$.

10. a^2bc, b^2ca, c^2ab.

11. $5a^2c$, $6cb^2$, $3bc^2$.

12. $2x^2y^3$, $3xy$, $4x^3y^4$.

13. $35a^2c^3b$, $42a^3cb^2$, $30ac^2b^3$.

14. $66a^4b^2c^3$, $44a^3b^4c^2$, $24a^2b^3c^4$.

15. $7a^2b$, $4ac^2$, $6ac^3$, $21bc$.

COMPOUND EXPRESSIONS.

128. L. C. M. of Compound Expressions which can be factored by Inspection. The method employed is similar to that of the preceding article.

Ex. 1. The lowest common multiple of $6x^2(a-x)^2$, $8a^3(a-x)^3$ and $12ax(a-x)^5$ is $24a^3x^2(a-x)^5$.

For it consists of the product of

(1) the numerical L. C. M. of the coefficients;

(2) the lowest power of each factor which is divisible by every power of that factor occurring in the given expressions.

Ex. 2. Find the lowest common multiple of

$$3a^2+9ab,\ 2a^3-18ab^2,\ a^3+6a^2b+9ab^2.$$

$$\begin{aligned} 3a^2+9ab &= 3a(a+3b), \\ 2a^3-18ab^2 &= 2a(a+3b)(a-3b), \\ a^3+6a^2b+9ab^2 &= a(a+3b)(a+3b) \\ &= a(a+3b)^2. \end{aligned}$$

Therefore the L. C. M. is $6a(a+3b)^2(a-3b)$.

EXAMPLES XII. b.

Find the lowest common multiple of

1. x^2, x^2-3x.

2. $21x^3$, $7x^2(x+1)$.

3. a^2+ab, $ab+b^2$.

4. $4x^2y-y$, $2x^2+x$.

5. $6x^2-2x$, $9x^2-3x$.

6. x^2+2x, x^2+3x+2.

7. x^2-3x+2, x^2-1.

8. $(a+x)^3$, a^3+x^3.

9. a^2+x^2, $(a+x)^2$.

10. $(a-x)^2$, a^2-x^2.

11. $(1+x)^3$, $1+x^3$.

12. x^2+4x+4, x^2+5x+6.

13. x^2-5x+4, x^2-6x+8.

14. $1-x^3$, $(1-x)^3$.

15. $(a-x)^3$, a^3-x^3.

16. x^2+x-20, $x^2-10x+24$, x^2-x-30.

17. x^2+x-42, $x^2-11x+30$, $x^2+2x-35$.

18. $2x^2+3x+1$, $2x^2+5x+2$, x^2+3x+2.

19. a^2-x^2, $(a-x)^2$, a^3-x^3.

20. $3x^2 + 11x + 6,\ 3x^2 + 8x + 4,\ x^2 + 5x + 6.$

21. $5x^2 + 11x + 2,\ 5x^2 + 16x + 3,\ x^2 + 5x + 6.$

22. $1 + x^2,\ (1 + x)^2,\ 1 + x^3.$

23. $2x^2 + 3x - 2,\ 2x^2 + 15x - 8,\ x^2 + 10x + 16.$

24. $3x^2 - x - 14,\ 3x^2 - 13x + 14,\ x^2 - 4.$

25. $12x^2 + 3x - 42,\ 12x^3 + 30x^2 + 12x,\ 32x^2 - 40x - 28.$

26. $3x^4 + 26x^3 + 35x^2,\ 6x^2 + 38x - 28,\ 27x^3 + 27x^2 - 30x.$

27. $60x^4 + 5x^3 - 5x^2,\ 60x^2y + 32xy + 4y,\ 40x^3y - 2x^2y - 2xy.$

28. $8x^2 - 38xy + 35y^2,\ 4x^2 - xy - 5y^2,\ 2x^2 - 5xy - 7y^2.$

29. $12x^2 - 23xy + 10y^2,\ 4x^2 - 9xy + 5y^2,\ 3x^2 - 5xy + 2y^2.$

30. $6ax^3 + 7a^2x^2 - 3a^3x,\ 3a^2x^2 + 14a^3x - 5a^4,\ 6x^2 + 39ax + 45a^2.$

31. $4ax^2y^2 + 11axy^2 - 3ay^2,\ 3x^3y^3 + 7x^2y^3 - 6xy^3,\ 24ax^2 - 22ax + 4a.$

129. L. C. M. of Compound Expressions which cannot be factored by Inspection. When the given expressions are such that their factors cannot be determined by inspection, they must be resolved by finding the highest common factor.

Ex. Find the lowest common multiple of

$$2x^4 + x^3 - 20x^2 - 7x + 24 \text{ and } 2x^4 + 3x^3 - 13x^2 - 7x + 15.$$

The highest common factor is $x^2 + 2x - 3$.
By division, we obtain

$$2x^4 + x^3 - 20x^2 - 7x + 24 = (x^2 + 2x - 3)(2x^2 - 3x - 8).$$
$$2x^4 + 3x^3 - 13x^2 - 7x + 15 = (x^2 + 2x - 3)(2x^2 - x - 5).$$

Therefore the L. C. M. is $(x^2+2x-3)(2x^2 - 3x-8)(2x^2-x-5)$.

130. We may now give the proof of the rule for finding the lowest common multiple of two compound algebraic expressions.

Let A and B be the two expressions, and F their highest common factor. Also suppose that a and b are the respective quotients when A and B are divided by F; then $A = aF$, $B = bF$. Therefore, since a and b have no common factor, the lowest common multiple of A and B is abF, by inspection.

131. There is an important relation between the highest common factor and the lowest common multiple of two expressions which it is desirable to notice.

Let F be the highest common factor, and X the lowest common multiple of A and B. Then, as in the preceding article,

$$A = aF,\ B = bF,$$

and
$$X = abF.$$

Therefore the product
$$\begin{aligned} AB &= aF \times bF \\ &= F \times abF \\ &= FX \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1). \end{aligned}$$

Hence *the product of two expressions is equal to the product of their highest common factor and lowest common multiple.*

Again, from (1) $X = \dfrac{AB}{F} = \dfrac{A}{F} \times B = \dfrac{B}{F} \times A$;

hence *the lowest common multiple of two expressions may be found by dividing their product by their highest common factor; or by dividing either of them by their highest common factor, and multiplying the quotient by the other.*

132. The lowest common multiple of three expressions A, B, C may be obtained as follows:

First find X, the L. C. M. of A and B. Next find Y, the L. C. M. of X and C; then Y will be the required L. C. M. of A, B, C.

For Y is the expression of lowest dimensions which is divisible by X and C, and X is the expression of lowest dimensions divisible by A and B. Therefore Y is the expression of lowest dimensions divisible by all three.

EXAMPLES XII. c.

1. Find the highest common factor and the lowest common multiple of $x^2 - 5x + 6,\ x^2 - 4,\ x^3 - 3x - 2$.

2. Find the lowest common multiple of
$$ab(x^2 + 1) + x(a^2 + b^2) \text{ and } ab(x^2 - 1) + x(a^2 - b^2).$$

3. Find the lowest common multiple of $xy - bx,\ xy - ay,$
$y^2 - 3by + 2b^2,\ xy - 2bx - ay + 2ab,\ xy - bx - ay + ab.$

4. Find the highest common factor and the lowest common multiple of $x^3 + 2x^2 - 3x$, $2x^3 + 5x^2 - 3x$.

5. Find the lowest common multiple of

$$1 - x,\ (1 - x^2)^2,\ (1 + x)^3.$$

6. Find the lowest common multiple of

$$x^2 - 10x + 24,\ x^2 - 8x + 12,\ x^2 - 6x + 8.$$

7. Find the highest common factor and the lowest common multiple of $6x^3 + x^2 - 5x - 2$, $6x^3 + 5x^2 - 3x - 2$.

8. Find the lowest common multiple of

$$(bc^2 - abc)^2,\ b^2(ac^2 - a^3),\ a^2c^2 + 2ac^3 + c^4.$$

9. Find the lowest common multiple of

$$x^3 - y^3,\ x^3y - y^4,\ y^2(x - y)^2,\ x^2 + xy + y^2.$$

Also find the highest common factor of the first three expressions.

10. Find the highest common factor of

$$6x^2 - 13x + 6,\ 2x^2 + 5x - 12,\ 6x^2 - x - 12.$$

Also show that the lowest common multiple is the product of the three quantities divided by the square of the highest common factor.

11. Find the lowest common multiple of

$$x^4 + ax^3 + a^3x + a^4,\ x^4 + a^2x^2 + a^4.$$

12. Find the highest common factor and the lowest common multiple of $3x^3 - 7x^2y + 5xy^2 - y^3$, $x^2y + 3xy^2 - 3x^3 - y^3$,

$$3x^3 + 5x^2y + xy^2 - y^3.$$

13. Find the highest common factor of

$$4x^3 - 10x^2 + 4x + 2,\ 3x^4 - 2x^3 - 3x + 2.$$

14. Find the lowest common multiple of

$$a^2 - b^2,\ a^3 - b^3,\ a^3 - a^2b - ab^2 - 2b^3.$$

15. Find the highest common factor and the lowest common multiple of $(2x^2 - 3a^2)y + (2a^2 - 3y^2)x$, $(2a^2 + 3y^2)x + (2x^2 + 3a^2)y$.

16. Find the highest common factor and the lowest common multiple of $x^3 - 9x^2 + 26x - 24$, $x^3 - 12x^2 + 47x - 60$.

17. Find the highest common factor of

$$x^3 - 15ax^2 + 48a^2x + 64a^3,\ x^2 - 10ax + 16a^2.$$

18. Find the lowest common multiple of

$$21x(xy - y^2)^2,\ 35(x^4y^2 - x^2y^4),\ 15y(x^2 + xy)^2.$$

CHAPTER XIII.

FRACTIONS.

133. DEFINITION. If a quantity x be divided into b equal parts, and a of these parts be taken, the result is called *the fraction* $\frac{a}{b}$ *of* x.

If x be the unit, the fraction $\frac{a}{b}$ of x is called simply "the fraction $\frac{a}{b}$"; so that *the fraction* $\frac{a}{b}$ *represents* a *equal parts,* b *of which make up the unit.*

The quantity above the horizontal line is spoken of as the numerator, and that below the line as the denominator of the fraction.

134. A **Simple Fraction** is one of which the numerator and denominator are whole numbers.

REDUCTION OF FRACTIONS.

135. *To reduce a fraction* is to change its form without changing its value.

136. To prove that $\frac{a}{b}=\frac{ma}{mb}$, where a, b, m are positive integers.

By $\frac{a}{b}$ we mean a equal parts, b of which make up the unit, (1);

by $\frac{ma}{mb}$ we mean ma equal parts, mb of which make up the unit, (2).

But $\qquad b$ parts in (1) $= mb$ parts in (2);

$\therefore$ 1 part in (1) $= m$ parts in (2);

$\therefore$ a parts in (1) $= ma$ parts in (2);

that is, $$\frac{a}{b} = \frac{ma}{mb}.$$

Conversely, $$\frac{ma}{mb} = \frac{a}{b}.$$

Hence, *The value of a fraction is not altered if we multiply or divide the numerator and denominator by the same quantity.*

137. Reduction of a Fraction to its Lowest Terms. As shown in the preceding article an algebraic fraction may be changed into an equivalent fraction by dividing numerator and denominator by any common factor; if this factor be the *highest common factor*, the resulting fraction is said to be **reduced to its lowest terms.**

Ex. 1. Reduce to lowest terms $\frac{24\,a^3c^2x^3}{36\,a^5x^2}$.

$$\frac{24\,a^3c^2x^3}{36\,a^5x^2} = \frac{2^3 \times 3\,a^3c^2x^3}{2^2 \times 3^2\,a^5x^2} = \frac{2\,c^2x}{3\,a^2}.$$

Ex. 2. Reduce to lowest terms $\frac{24\,a^3c^2x^2}{18\,a^3x^2 - 12\,a^2x^3}$.

$$\frac{24\,a^3c^2x^2}{18a^3x^2 - 12\,a^2x^3} = \frac{24\,a^3c^2x^2}{6\,a^2x^2(3\,a - 2\,x)} = \frac{4\,ac^2}{3\,a - 2\,x}.$$

Ex. 3. Reduce to lowest terms $\frac{6\,x^2 - 8\,xy}{9\,xy - 12\,y^2}$.

$$\frac{6\,x^2 - 8\,xy}{9\,xy - 12\,y^2} = \frac{2\,x(3\,x - 4\,y)}{3\,y(3\,x - 4\,y)} = \frac{2\,x}{3\,y}.$$

NOTE. The beginner should be careful not to begin cancelling until he has expressed both numerator and denominator in the most convenient form, by resolution into factors where necessary.

EXAMPLES XIII. a.

Reduce to lowest terms:

1. $\frac{12\,mn^2p}{15\,m^2np^2}$.
2. $\frac{46\,x^3y^4z^5}{69\,x^2y^3z^4}$.
3. $\frac{ax}{a^2x^2 - ax}$.
4. $\frac{3\,a^2 - 6\,ab}{2\,a^2b - 4\,ab^2}$.
5. $\frac{abx + bx^2}{acx + cx^2}$.
6. $\frac{15\,a^2b^2c}{100(a^3 - a^2b)}$.

7. $\dfrac{4x^2-9y^2}{4x^2+6xy}$.

8. $\dfrac{20(x^3-y^3)}{5x^2+5xy+5y^2}$.

9. $\dfrac{x(2a^2-3ax)}{a(4a^2x-9x^3)}$.

10. $\dfrac{x^3-2xy^2}{x^4-4x^2y^2+4y^4}$.

11. $\dfrac{(xy-3y^2)^2}{x^3y^2-27y^5}$.

12. $\dfrac{x^2-5x}{x^2-4x-5}$.

13. $\dfrac{3x^2+6x}{x^2+4x+4}$.

14. $\dfrac{5a^3b+10a^2b^2}{3a^2b^2+6ab^3}$.

15. $\dfrac{x^3y+2x^2y+4xy}{x^3-8}$.

16. $\dfrac{3a^4+9a^3b+6a^2b^2}{a^4+a^3b-2a^2b^2}$.

17. $\dfrac{x^4-14x^2-51}{x^4-2x^2-15}$.

18. $\dfrac{x^2+xy-2y^2}{x^3-y^3}$.

19. $\dfrac{2x^2+17x+21}{3x^2+26x+35}$.

20. $\dfrac{a^2x^2-16a^2}{ax^2+9ax+20a}$.

21. $\dfrac{3x^2+23x+14}{3x^2+41x+26}$.

22. $\dfrac{27a+a^4}{18a-6a^2+2a^3}$.

138. When the factors of the numerator and denominator cannot be determined by inspection, we find the highest common factor, by the rules given in Chapter XI.

Ex. Reduce to lowest terms $\dfrac{3x^3-13x^2+23x-21}{15x^3-38x^2-2x+21}$.

First Method. The H. C. F. of numerator and denominator is $3x-7$.

Dividing numerator and denominator by $3x-7$, we obtain as respective quotients x^2-2x+3 and $5x^2-x-3$.

$$\text{Thus}\quad \frac{3x^3-13x^2+23x-21}{15x^3-38x^2-2x+21}=\frac{(3x-7)(x^2-2x+3)}{(3x-7)(5x^2-x-3)}=\frac{x^2-2x+3}{5x^2-x-3}.$$

This is the simplest solution for the beginner; but in this and similar cases we may often effect the reduction without actually going through the process of finding the highest common factor.

Second Method. By Art. 116, the H. C. F. of numerator and denominator must be a factor of their sum $18x^3-51x^2+21x$, that is, of $3x(3x-7)(2x-1)$. If there be a common divisor it must clearly be $3x-7$; hence arranging numerator and denominator so as to show $3x-7$ as a factor,

$$\text{the fraction}=\frac{x^2(3x-7)-2x(3x-7)+3(3x-7)}{5x^2(3x-7)-x(3x-7)-3(3x-7)}$$

$$=\frac{(3x-7)(x^2-2x+3)}{(3x-7)(5x^2-x-3)}=\frac{x^2-2x+3}{5x^2-x-3}.$$

139. If either numerator or denominator can readily be resolved into factors we may use the following method.

Ex. Reduce to lowest terms $\dfrac{x^3+3x^2-4x}{7x^3-18x^2+6x+5}$.

The numerator $= x(x^2+3x-4) = x(x+4)(x-1)$.

Of these factors the only one which can be a common divisor is $x-1$. Hence, arranging the denominator,

$$\text{the fraction } = \frac{x(x+4)(x-1)}{7x^2(x-1)-11x(x-1)-5(x-1)}$$

$$= \frac{x(x+4)(x-1)}{(x-1)(7x^2-11x-5)} = \frac{x(x+4)}{7x^2-11x-5}.$$

EXAMPLES XIII. b.

Reduce to lowest terms:

1. $\dfrac{a^3-a^2b-ab^2-2b^3}{a^3+3a^2b+3ab^2+2b^3}$.

2. $\dfrac{x^3-5x^2+7x-3}{x^3-3x+2}$.

3. $\dfrac{a^3+2a^2-13a+10}{a^3+a^2-10a+8}$.

4. $\dfrac{2x^3+5x^2y-30xy^2+27y^3}{4x^3+5xy^2-21y^3}$.

5. $\dfrac{4a^3+12a^2b-ab^2-15b^3}{6a^3+13a^2b-4ab^2-15b^3}$.

6. $\dfrac{1+2x^2+x^3+2x^4}{1+3x^2+2x^3+3x^4}$.

7. $\dfrac{x^2-2x+1}{3x^3+7x-10}$.

8. $\dfrac{3a^3-3a^2b+ab^2-b^3}{4a^2-5ab+b^2}$.

9. $\dfrac{4x^3+3ax^2+a^3}{x^4+ax^3+a^3x+a^4}$.

10. $\dfrac{4x^3-10x^2+4x+2}{3x^4-2x^3-3x+2}$.

11. $\dfrac{16x^4-72x^2a^2+81a^4}{4x^2+12ax+9a^2}$.

12. $\dfrac{6x^3+x^2-5x-2}{6x^3+5x^2-3x-2}$.

13. $\dfrac{5x^3+2x^2-15x-6}{7x^3-4x^2-21x+12}$.

14. $\dfrac{4x^4+11x^2+25}{4x^4-9x^2+30x-25}$.

15. $\dfrac{3x^3-27ax^2+78a^2x-72a^3}{2x^3+10ax^2-4a^2x-48a^3}$.

16. $\dfrac{ax^3-5a^2x^2-99a^3x+40a^4}{x^4-6ax^3-86a^2x^2+35a^3x}$.

MULTIPLICATION AND DIVISION OF FRACTIONS.

140. Rule I. To multiply a fraction by an integer. *Multiply the numerator by that integer; or, if the denominator be divisible by the integer, divide the denominator by it.*

The proof is as follows:

(1) $\frac{a}{b}$ represents a equal parts, b of which make up the unit; $\frac{ac}{b}$ represents ac equal parts, b of which make up the unit; and the number of parts taken in the second fraction is c times the number taken in the first;

that is, $$\frac{a}{b} \times c = \frac{ac}{b}.$$

(2) $$\frac{a}{bd} \times d = \frac{ad}{bd} = \frac{a}{b}.$$ [Art. 136.]

Hence $\frac{a}{b} \times b = \frac{ab}{b} = a$; that is, the fraction $\frac{a}{b}$ is the quantity which must be multiplied by b in order to obtain a. Now the quantity which must be multiplied by b in order to obtain a is the quotient resulting from the division of a by b [Art. 53]; therefore we may define a fraction thus: *the fraction* $\frac{a}{b}$ *is the quotient of* a *divided by* b.

141. Rule II. To divide a fraction by an integer. *Divide the numerator, if it be divisible, by the integer; or, if the numerator be not divisible, multiply the denominator by that integer.*

The proof is as follows:

(1) $\frac{ac}{b}$ represents ac equal parts, b of which make up the unit; $\frac{a}{b}$ represents a equal parts, b of which make up the unit.

The number of parts taken in the first fraction is c times the number taken in the second. Therefore the second fraction is the quotient of the first fraction divided by c;

that is, $$\frac{ac}{b} \div c = \frac{a}{b}.$$

(2) But if the numerator be not divisible by c, we have

$$\frac{a}{b}=\frac{ac}{bc};$$

$$\therefore\ \frac{a}{b}\div c=\frac{ac}{bc}\div c=\frac{a}{bc}, \text{ by the preceding case.}$$

142. Rule III. To multiply together two or more fractions. *Multiply the numerators for a new numerator, and the denominators for a new denominator.*

To find the value of $\frac{a}{b}\times\frac{c}{d}$.

Let $$x=\frac{a}{b}\times\frac{c}{d}.$$

Multiplying each side by $b\times d$, we have

$$x\times b\times d=\frac{a}{b}\times\frac{c}{d}\times b\times d$$

$$=\frac{a}{b}\times b\times\frac{c}{d}\times d \qquad \text{[Art. 37.]}$$

$$=a\times c. \qquad \text{[Art. 140.]}$$

$$\therefore\ xbd=ac.$$

Dividing each side by bd, we have

$$x=\frac{ac}{bd};$$

$$\therefore\ \frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}.$$

Similarly, $\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}=\frac{ace}{bdf}$; and so for any number of fractions.

143. Rule IV. To divide one fraction by another. *Invert the divisor, and proceed as in multiplication.*

Since division is the inverse of multiplication, we may define the quotient x, when $\frac{a}{b}$ is divided by $\frac{c}{d}$, to be such that

$$x\times\frac{c}{d}=\frac{a}{b}.$$

Multiplying by $\frac{d}{c}$, we have $x \times \frac{c}{d} \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c}$;

$$\therefore\ x = \frac{ad}{bc}.$$

Hence $$\frac{a}{b} \div \frac{c}{d} = \frac{ad}{bc} = \frac{a}{b} \times \frac{d}{c}. \qquad \text{[Art. 142.]}$$

Ex. 1. Simplify $\frac{2a}{3b} \times \frac{3c^2}{4a^3} \times \frac{5bc}{6abc}$.

The expression $= \frac{2 \times 3 \times 5 \times abc^3}{3 \times 4 \times 6 \times a^4b^2c} = \frac{5c^2}{12a^3b}$.

Ex. 2. Simplify $\frac{2a^2 + 3a}{4a^3} \times \frac{4a^2 - 6a}{12a + 18}$.

$$\frac{2a^2 + 3a}{4a^3} \times \frac{4a^2 - 6a}{12a + 18} = \frac{a(2a + 3)}{4a^3} \times \frac{2a(2a - 3)}{6(2a + 3)} = \frac{2a - 3}{12a},$$

by cancelling those factors which are common to both numerator and denominator.

Ex. 3. Simplify $\frac{6x^2 - ax - 2a^2}{ax - a^2} \times \frac{x - a}{9x^2 - 4a^2} \div \frac{2x + a}{3ax + 2a^2}$.

The expression

$$= \frac{6x^2 - ax - 2a^2}{ax - a^2} \times \frac{x - a}{9x^2 - 4a^2} \times \frac{3ax + 2a^2}{2x + a}$$

$$= \frac{(3x - 2a)(2x + a)}{a(x - a)} \times \frac{x - a}{(3x + 2a)(3x - 2a)} \times \frac{a(3x + 2a)}{2x + a} = 1,$$

since all the factors cancel each other.

EXAMPLES XIII. c.

Simplify

1. $\frac{7a^2b^3}{9ax^2y} \times \frac{18x^2c}{15ac^4}$.

2. $\frac{21k^2p^3}{13mn^2} \div \frac{28p^2k^3}{39m^2n^3}$.

3. $\frac{2x^2y}{3yz} \times \frac{5z^2x}{7xy^2} \div \frac{21x^2y^3z^2}{40xy^2z}$.

4. $\frac{m^2}{8n} \times \frac{36p^3q^2}{81mn} \div \frac{15mpx^5}{27n^2x^3y}$.

5. $\frac{14x^2 - 7x}{12x^3 + 24x^2} \div \frac{2x - 1}{x^2 + 2x}$.

6. $\frac{a^2b^2 + 3ab}{4a^2 - 1} \div \frac{ab + 3}{2a + 1}$.

7. $\frac{x^2 - 4a^2}{ax + 2a^2} \times \frac{2a}{x - 2a}$.

8. $\frac{a^2 - 121}{a^2 - 4} \div \frac{a + 11}{a + 2}$.

9. $\dfrac{16x^2-9a^2}{x^2-4}\times\dfrac{x-2}{4x-3a}$.

10. $\dfrac{25a^2-b^2}{9a^2x^2-4x^2}\times\dfrac{x(3a+2)}{5a+b}$.

11. $\dfrac{x^2+5x+6}{x^2-1}\times\dfrac{x^2-2x-3}{x^2-9}$.

12. $\dfrac{x^2+3x+2}{x^2+9x+20}\times\dfrac{x^2+7x+12}{x^2+5x+6}$.

13. $\dfrac{2x^2+5x+2}{x^2-4}\times\dfrac{x^2+4x}{2x^2+9x+4}$.

14. $\dfrac{b^4-27b}{2b^2+5b}\times\dfrac{4b^2-25}{2b^2-11b+15}$.

15. $\dfrac{2x^2+13x+15}{4x^2-9}\div\dfrac{2x^2+11x+5}{4x^2-1}$.

16. $\dfrac{3a^2+3ax}{(a-x)^2}\times\dfrac{2a^2+ax-3x^2}{a^2-x^2}$.

17. $\dfrac{x^2-14x-15}{x^2-4x-45}\div\dfrac{x^2-12x-45}{x^2-6x-27}$.

18. $\dfrac{2+5x+3x^2}{(1+x)^3}\div\dfrac{4x^2}{1+x^3}$.

19. $\dfrac{2x^2-x-1}{2x^2+5x+2}\times\dfrac{4x^2+x-14}{16x^2-49}$.

20. $\dfrac{a^2-2ab-3b^2}{(a+b)^3}\div\dfrac{a^2-4ab+3b^2}{a^3+b^3}$.

21. $\dfrac{x^3-6x^2+36x}{x^2-49}\div\dfrac{x^4+216x}{x^2-x-42}$.

22. $\dfrac{64p^2q^2-z^4}{x^2-4}\times\dfrac{(x-2)^2}{8pq+z^2}\div\dfrac{x^2-4}{(x+2)^2}$.

23. $\dfrac{a^2-2a-3}{(a+3)^2}\times\dfrac{a^2-a-12}{a^2+9}$.

24. $\dfrac{x^2-x-20}{x^2-25}\times\dfrac{x^2-x-2}{x^2+2x-8}\div\dfrac{x+1}{x^2+5x}$.

25. $\dfrac{x^2-18x+80}{x^2-5x-50}\times\dfrac{x^2-6x-7}{x^2-15x+56}\times\dfrac{x+5}{x-1}$.

26. $\dfrac{1-x^2}{1-x^3}\times\dfrac{1-3x^2+2x^3}{(1-x)^3}$.

27. $\dfrac{x^2-5x-14}{(x-2)^2}\times\dfrac{x^2-11x+18}{x^2-4}\div\dfrac{x^2-16x+63}{ax^3-4a^2x+4ax}$.

ADDITION AND SUBTRACTION OF FRACTIONS.

144. To prove $\quad \dfrac{a}{b}+\dfrac{c}{d}=\dfrac{ad+bc}{bd}$.

We have $\quad \dfrac{a}{b}=\dfrac{ad}{bd}$, and $\dfrac{c}{d}=\dfrac{bc}{bd}$. [Art. 136.]

Thus in each case we divide the unit into bd equal parts, and we take first ad of these parts, and then bc of them; that is, we take $ad + bc$ of the bd parts of the unit; and this is expressed by the fraction $\frac{ad + bc}{bd}$.

$$\therefore \quad \frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

Similarly, $$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

145. Here the fractions have been both expressed with a common denominator bd. But if b and d have a common factor, the product bd is not the lowest common denominator, and the fraction $\frac{ad + bc}{bd}$ *will not be in its lowest terms.* To avoid working with fractions which are not in their lowest terms, we take the *lowest* common denominator, which is the lowest common multiple of the denominators of the given fractions.

Rule I. To reduce fractions to their lowest common denominator. *Find the L. C. M. of the given denominators, and take it for the common denominator; divide it by the denominator of the first fraction, and multiply the numerator of this fraction by the quotient so obtained; and do the same with all the other given fractions.*

Ex. 1. Express with lowest common denominator

$$\frac{a}{3\,xy}, \quad \frac{b}{6\,xyz}, \quad \frac{c}{2\,yz}.$$

The lowest common multiple of the denominators is $6\,xyz$. Dividing this by each of the denominators in turn, and multiplying the corresponding numerators by the respective quotients, we have the equivalent fractions

$$\frac{2\,az}{6\,xyz}, \quad \frac{b}{6\,xyz}, \quad \frac{3\,cx}{6\,xyz}.$$

Ex. 2. Express with lowest common denominator

$$\frac{5\,x}{2\,a(x-a)} \text{ and } \frac{4\,a}{3\,x(x^2-a^2)}.$$

The lowest common denominator is $6ax(x-a)(x+a)$.

We must, therefore, multiply the numerators by $3x(x+a)$ and $2a$ respectively.

Hence the equivalent fractions are

$$\frac{15x^2(x+a)}{6ax(x-a)(x+a)} \text{ and } \frac{8a^2}{6ax(x-a)(x+a)}.$$

146. We may now enunciate the rule for the addition or subtraction of fractions.

Rule II. To add or subtract fractions. *Reduce them to the lowest common denominator; add or subtract the numerators, and retain the common denominator.*

Ex. 1. Find the value of $\frac{2x+a}{3a}+\frac{5x-4a}{9a}$.

The lowest common denominator is $9a$.

Therefore the expression $=\frac{3(2x+a)+5x-4a}{9a}$

$$=\frac{6x+3a+5x-4a}{9a}=\frac{11x-a}{9a}.$$

Ex. 2. Find the value of $\frac{x-2y}{xy}+\frac{3y-a}{ay}-\frac{3x-2a}{ax}$.

The lowest common denominator is axy.

Thus the expression $=\frac{a(x-2y)+x(3y-a)-y(3x-2a)}{axy}$

$$=\frac{ax-2ay+3xy-ax-3xy+2ay}{axy}=0,$$

since the terms in the numerator destroy each other.

EXAMPLES XIII. d.

Find the value of

1. $\frac{2x-1}{3}+\frac{x-5}{6}+\frac{x-4}{4}$.

2. $\frac{2x-3}{9}+\frac{x+2}{6}+\frac{5x+8}{12}$.

3. $\frac{x-7}{15}+\frac{x-9}{25}-\frac{x+3}{45}$.

4. $\frac{2x+5}{x}-\frac{x+3}{2x}-\frac{27}{8x^2}$.

5. $\frac{a-b}{ab}+\frac{b-c}{bc}+\frac{c-a}{ca}$.

6. $\frac{a-2b}{2a}-\frac{a-5b}{4a}+\frac{a+7b}{8a}$.

7. $\frac{a-x}{x}+\frac{a+x}{a}-\frac{a^2-x^2}{2ax}$.

8. $\frac{2a^2-b^2}{a^2}-\frac{b^2-c^2}{b^2}-\frac{c^2-a^2}{c^2}$.

9. $\frac{x-3}{5x}+\frac{x^2-9}{10x^2}-\frac{8-x^3}{15x^3}$

10. $\frac{2}{xy}-\frac{3y^2-x^2}{xy^3}+\frac{xy+y^2}{x^2y^2}$.

11. $\frac{2x-3y}{xy}+\frac{3x-2z}{xz}+\frac{5}{x}$.

12. $\frac{a^2-bc}{bc}-\frac{ac-b^2}{ac}-\frac{ab-c^2}{ab}$.

Ex. 3. Simplify $\frac{2x-3a}{x-2a}-\frac{2x-a}{x-a}$.

The lowest common denominator is $(x-2a)(x-a)$.

$$\begin{aligned}\text{Therefore the expression} &= \frac{(2x-3a)(x-a)-(2x-a)(x-2a)}{(x-2a)(x-a)}\\ &= \frac{2x^2-5ax+3a^2-(2x^2-5ax+2a^2)}{(x-2a)(x-a)}\\ &= \frac{2x^2-5ax+3a^2-2x^2+5ax-2a^2}{(x-2a)(x-a)}\\ &= \frac{a^2}{(x-2a)(x-a)}.\end{aligned}$$

Note. In finding the value of such an expression as

$$-(2x-a)(x-2a),$$

the beginner should first express the product in brackets, and then remove the brackets, as we have done. After a little practice he will be able to take both steps together.

The work will sometimes be shortened by first reducing the fractions to their lowest terms.

Ex. 4. Simplify $\frac{x^2+5xy-4y^2}{x^2-16y^2}-\frac{2xy}{2x^2+8xy}$.

$$\begin{aligned}\text{The expression} &= \frac{x^2+5xy-4y^2}{x^2-16y^2}-\frac{y}{x+4y}\\ &= \frac{x^2+5xy-4y^2-y(x-4y)}{x^2-16y^2}\\ &= \frac{x^2+5xy-4y^2-xy+4y^2}{x^2-16y^2}\\ &= \frac{x^2+4xy}{x^2-16y^2}=\frac{x}{x-4y}.\end{aligned}$$

EXAMPLES XIII. e.

Find the value of

1. $\dfrac{1}{x+2}+\dfrac{1}{x+3}$.

2. $\dfrac{2}{x+3}-\dfrac{1}{x+4}$.

3. $\dfrac{3}{x-6}-\dfrac{1}{x+2}$.

4. $\dfrac{a}{x+a}-\dfrac{b}{x+b}$.

5. $\dfrac{a}{x-a}+\dfrac{b}{x-b}$.

6. $\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}$.

7. $\dfrac{x+2}{x-2}-\dfrac{x-2}{x+2}$.

8. $\dfrac{x-4}{x-2}-\dfrac{x-7}{x-5}$.

9. $\dfrac{a}{x-a}-\dfrac{a^2}{x^2-a^2}$.

10. $\dfrac{a}{x^2-4}+\dfrac{b}{(x-2)^2}$.

11. $\dfrac{3}{x-3}+\dfrac{2x}{x^2-9}$.

12. $\dfrac{1}{2x-3y}-\dfrac{x+y}{4x^2-9y^2}$.

13. $\dfrac{1}{1-x^3}-\dfrac{1}{(1-x)^3}$.

14. $\dfrac{x+a}{x-2a}-\dfrac{x^2+2a^2}{x^2-4a^2}$.

15. $\dfrac{4a^2+b^2}{4a^2-b^2}-\dfrac{2a-b}{2a+b}$.

16. $\dfrac{2x^2}{x^2-y^2}-\dfrac{2x}{x+y}$.

17. $\dfrac{x}{1-x^2}-\dfrac{x}{1+x^2}$.

18. $\dfrac{1}{x(x-y)}+\dfrac{1}{y(x+y)}$.

19. $\dfrac{2}{a(x^2-a^2)}-\dfrac{2}{x(x+a)^2}$.

20. $\dfrac{xy}{25x^2-y^2}+\dfrac{x}{5x+y}$.

21. $\dfrac{y}{x(x^2-y^2)}+\dfrac{x}{y(x^2+y^2)}$.

22. $\dfrac{x+a}{(x^2+a^2)}-\dfrac{x-a}{(x+a)^2}$.

23. $\dfrac{x^2-4a^2}{x^2-2ax}-\dfrac{x+4a}{x+2a}$.

24. $\dfrac{x^2+xy+y^2}{x+y}+\dfrac{x^2-xy+y^2}{x-y}$.

25. $\dfrac{1}{a-2x}-\dfrac{(a+2x)^2}{a^3-8x^3}$.

26. $\dfrac{a^3+b^3}{a^2-ab+b^2}-\dfrac{a^3-b^3}{a^2+ab+b^2}$.

147. Some modification of the foregoing general methods may sometimes be used with advantage. The most useful artifices are explained in the examples which follow, but no general rules can be given which will apply to all cases.

Ex. 1. Simplify $\dfrac{a+3}{a-4}-\dfrac{a+4}{a-3}-\dfrac{8}{a^2-16}$.

Taking the first two fractions together, we have the expression

$$=\frac{a^2-9-(a^2-16)}{(a-4)(a-3)}-\frac{8}{a^2-16}=\frac{7}{(a-4)(a-3)}-\frac{8}{(a+4)(a-4)}$$

$$=\frac{7(a+4)-8(a-3)}{(a+4)(a-4)(a-3)}=\frac{52-a}{(a+4)(a-4)(a-3)}.$$

Ex. 2. Simplify $\dfrac{1}{2x^2+x-1}+\dfrac{1}{3x^2+4x+1}$.

The expression
$$=\frac{1}{(2x-1)(x+1)}+\frac{1}{(3x+1)(x+1)}$$
$$=\frac{3x+1+2x-1}{(2x-1)(x+1)(3x+1)}$$
$$=\frac{5x}{(2x-1)(x+1)(3x+1)}.$$

Ex. 3. Simplify $\dfrac{1}{a-x}-\dfrac{1}{a+x}-\dfrac{2x}{a^2+x^2}-\dfrac{4x^3}{a^4+x^4}$.

Here it should be evident that the first two denominators give L. C. M. a^2-x^2, which readily combines with a^2+x^2 to give L. C. M. a^4-x^4, which again combines with a^4+x^4 to give L. C. M. a^8-x^8. Hence it will be convenient to proceed as follows:

The expression
$$=\frac{a+x-(a-x)}{a^2-x^2}-\dots-\dots$$
$$=\frac{2x}{a^2-x^2}-\frac{2x}{a^2+x^2}-\dots$$
$$=\frac{4x^3}{a^4-x^4}-\frac{4x^3}{a^4+x^4}=\frac{8x^7}{a^8-x^8}.$$

EXAMPLES XIII. f.

Find the value of

1. $\dfrac{1}{x+y}-\dfrac{1}{x-y}+\dfrac{2x}{x^2-y^2}$.

2. $\dfrac{1}{2x+y}+\dfrac{1}{2x-y}-\dfrac{3x}{4x^2-y^2}$.

3. $\dfrac{5}{1+2x}-\dfrac{3x}{1-2x}-\dfrac{4-13x}{1-4x^2}$.

4. $\dfrac{2a}{2a+3b}+\dfrac{3b}{2a-3b}-\dfrac{8b^2}{4a^2-9b^2}$.

5. $\dfrac{1}{a+x}+\dfrac{1}{(a+x)^2}-\dfrac{1}{a^2-x^2}$.

6. $\dfrac{10}{9-a^2}-\dfrac{2}{3+a}-\dfrac{1}{3-a}$.

7. $\dfrac{5x}{6(x^2-1)}-\dfrac{1}{2(x-1)}+\dfrac{1}{3(x+1)}$.

8. $\dfrac{1}{2(a-b)}-\dfrac{1}{2(a+b)}-\dfrac{b}{a^2-b^2}$.

9. $\dfrac{1}{1+x}+\dfrac{2}{1-x^2}+\dfrac{3}{(1+x)^2}$.

10. $\dfrac{2a}{2a-3}-\dfrac{5}{6a+9}-\dfrac{4(3a+2)}{3(4a^2-9)}$.

11. $\dfrac{3}{x-2}+\dfrac{2}{3x+6}+\dfrac{5x}{x^2-4}$.

12. $\dfrac{x}{x^3+y^3}-\dfrac{y}{x^3-y^3}+\dfrac{x^3y+xy^3}{x^6-y^6}$.

13. $\dfrac{1}{x^2-9x+20}+\dfrac{1}{x^2-11x+30}$.

14. $\dfrac{1}{x^2-7x+12}-\dfrac{1}{x^2-5x+6}$.

15. $\dfrac{1}{2x^2-x-1}-\dfrac{1}{2x^2+x-3}$.

17. $\dfrac{4}{4-7a-2a^2}-\dfrac{3}{3-a-10a^2}$.

16. $\dfrac{1}{2x^2-x-1}-\dfrac{3}{6x^2-x-2}$.

18. $\dfrac{5}{5+x-18x^2}-\dfrac{2}{2+5x+2x^2}$.

19. $\dfrac{1}{x+1}-\dfrac{1}{(x+1)(x+2)}+\dfrac{1}{(x+1)(x+2)(x+3)}$.

20. $\dfrac{5x}{2(x+1)(x-3)}-\dfrac{15(x-1)}{16(x-3)(x-2)}-\dfrac{9(x+3)}{16(x+1)(x-2)}$.

21. $\dfrac{a+3b}{4(a+b)(a+2b)}+\dfrac{a+2b}{(a+b)(a+3b)}-\dfrac{a+b}{4(a+2b)(a+3b)}$.

22. $\dfrac{2}{x^2-3x+2}+\dfrac{2}{x^2-x-2}-\dfrac{1}{x^2-1}$.

23. $\dfrac{x}{x^2+5x+6}+\dfrac{15}{x^2+9x+14}-\dfrac{12}{x^2+10x+21}$.

24. $\dfrac{3}{x^2-1}+\dfrac{4}{2x+1}+\dfrac{4x+2}{2x^2+3x+1}$.

25. $\dfrac{5(2x-3)}{11(6x^2+x-1)}+\dfrac{7x}{6x^2+7x-3}-\dfrac{12(3x+1)}{11(4x^2+8x+3)}$.

26. $\dfrac{x-3}{x+2}-\dfrac{x-2}{x+3}+\dfrac{1}{x-1}$.

28. $\dfrac{1+2a}{1-2a}-\dfrac{1-2a}{1+2a}-\dfrac{8a}{(1-2a)^2}$.

27. $\dfrac{x-3}{x-4}-\dfrac{x+4}{x+3}-\dfrac{5}{x^2-16}$.

29. $\dfrac{1}{1+x}-\dfrac{x}{1+x^3}-\dfrac{x^2}{(1-x)^3}$.

30. $\dfrac{24x}{9-12x+4x^2}-\dfrac{3+2x}{3-2x}+\dfrac{3-2x}{3+2x}$.

31. $\dfrac{1}{3-x}-\dfrac{1}{3+x}-\dfrac{2x}{9+x^2}$.

32. $\dfrac{1}{2a+3}+\dfrac{1}{2a-3}-\dfrac{4a}{4a^2+9}$.

33. $\dfrac{1}{4(1+x)}+\dfrac{1}{4(1-x)}+\dfrac{1}{2(1+x^2)}$.

34. $\dfrac{3}{8(a-x)}+\dfrac{1}{8(a+x)}-\dfrac{a-x}{4(a^2+x^2)}$.

35. $\dfrac{2x}{4+x^2}+\dfrac{1}{2-x}-\dfrac{1}{2+x}$.

36. $\dfrac{5}{3-6x}-\dfrac{5}{3+6x}-\dfrac{x}{2+8x^2}$.

37. $\dfrac{1}{2a-8x}-\dfrac{a}{3a^2+48x^2}+\dfrac{1}{2a+8x}$.

38. $\dfrac{1}{6a^2+54}+\dfrac{1}{3a-9}-\dfrac{a}{3a^2-27}$.

39. $\dfrac{2x}{2+x}-\dfrac{4x}{(2+x)^2}-\dfrac{2x^2}{4-x^2}$.

40. $\dfrac{1}{8-8x}-\dfrac{1}{8+8x}+\dfrac{x}{4+4x^2}-\dfrac{x}{2+2x^4}$.

148. We have thus far assumed both numerator and denominator to be positive integers, and have shown in Art. 140 that a fraction itself is the quotient resulting from the division of numerator by denominator. But in algebra division is a process not restricted to positive integers, and we shall now extend this definition as follows: *The algebraic fraction* $\frac{a}{b}$ *is the quotient resulting from the division of a by b, where a and b may have any values whatever.*

149. By the preceding article $\frac{-a}{-b}$ is the quotient resulting from the division of $-a$ by $-b$; and this is obtained by dividing a by b, and, by the rule of signs, prefixing $+$.

Therefore $$\frac{-a}{-b} = +\frac{a}{b} = \frac{a}{b} \quad \ldots\ldots\ldots \quad (1).$$

Again, $\frac{-a}{b}$ is the quotient resulting from the division of $-a$ by b; and this is obtained by dividing a by b, and, by the rule of signs, prefixing $-$.

Therefore $$\frac{-a}{b} = -\frac{a}{b} \quad \ldots\ldots\ldots \quad (2).$$

Similarly, $$\frac{a}{-b} = -\frac{a}{b} \quad \ldots\ldots\ldots \quad (3).$$

These results may be enunciated as follows:

(1) *If the signs of* BOTH *numerator and denominator of a fraction be changed, the sign of the whole fraction will be unchanged.*

(2) *If the sign of* EITHER *numerator or denominator alone be changed, the sign of the whole fraction will be changed.*

The principles here involved are so useful in certain cases of reduction of fractions that we quote them in another form, which will sometimes be found more easy of application.

1. *We may change the sign of every term in the numerator and denominator of a fraction without altering its value.*

2. *We may change the sign of a fraction by simply changing the sign of every term in* EITHER *the numerator or denominator.*

NOTE. The student should keep clearly in his mind the distinction between *term* and *factor*. The rule governing change of sign for factors will be given in Art. 150.

Ex. 1. $$\frac{b-a}{y-x}=\frac{-b+a}{-y+x}=\frac{a-b}{x-y}.$$

Ex. 2. $$\frac{x-x^2}{2y}=-\frac{-x+x^2}{2y}=-\frac{x^2-x}{2y}.$$

Ex. 3. $$\frac{3x}{4-x^2}=-\frac{3x}{-4+x^2}=-\frac{3x}{x^2-4}.$$

The intermediate step may usually be omitted.

Ex. 4. Simplify $\dfrac{a}{x+a}+\dfrac{2x}{x-a}+\dfrac{a(3x-a)}{a^2-x^2}$.

Here it is evident that the lowest common denominator of the first two fractions is x^2-a^2, therefore it will be convenient to alter the sign of the denominator in the third fraction.

Thus the expression
$$=\frac{a}{x+a}+\frac{2x}{x-a}-\frac{a(3x-a)}{x^2-a^2}$$
$$=\frac{a(x-a)+2x(x+a)-a(3x-a)}{x^2-a^2}$$
$$=\frac{ax-a^2+2x^2+2ax-3ax+a^2}{x^2-a^2}=\frac{2x^2}{x^2-a^2}.$$

Ex. 5. Simplify $\dfrac{5}{3x-3}+\dfrac{3x-1}{1-x^2}+\dfrac{1}{2x+2}$.

The expression
$$=\frac{5}{3(x-1)}-\frac{3x-1}{x^2-1}+\frac{1}{2(x+1)}$$
$$=\frac{10(x+1)-6(3x-1)+3(x-1)}{6(x^2-1)}$$
$$=\frac{10x+10-18x+6+3x-3}{6(x^2-1)}=\frac{13-5x}{6(x^2-1)}.$$

EXAMPLES XIII. g.

Simplify

1. $\dfrac{1}{4x-4}-\dfrac{1}{5x+5}+\dfrac{1}{1-x^2}$.

2. $\dfrac{3}{1+a}-\dfrac{2}{1-a}-\dfrac{5a}{a^2-1}$.

3. $\dfrac{x+2a}{x+a}+\dfrac{2(a^2-4ax)}{a^2-x^2}-\dfrac{3a}{x-a}$.

4. $\dfrac{x-a}{x+a}+\dfrac{a^2+3ax}{a^2-x^2}+\dfrac{x+a}{x-a}$.

5. $\dfrac{1}{2x+1}+\dfrac{1}{2x-1}+\dfrac{4x}{1-4x^2}$.

6. $\dfrac{3x}{1-x^2}-\dfrac{2}{x-1}-\dfrac{2}{x+1}$.

7. $\dfrac{2-5x}{x+3}-\dfrac{3+x}{3-x}+\dfrac{2x(2x-11)}{x^2-9}$.

8. $\dfrac{3-2x}{2x+3}-\dfrac{2x+3}{3-2x}+\dfrac{12}{4x^2-9}$.

9. $\dfrac{5}{2b+2}-\dfrac{3}{4b-4}+\dfrac{11}{6-6b^2}$.

10. $\dfrac{1}{6a+6}+\dfrac{1}{6-6a}-\dfrac{1}{3a^2-3}$.

11. $\dfrac{y^2}{x^3-y^3}+\dfrac{x^3y^2}{y^6-x^6}$.

12. $\dfrac{x^2-y^2}{xy}-\dfrac{xy-y^2}{xy-x^2}$.

13. $\dfrac{x^2+y^2}{x^2-y^2}+\dfrac{x}{x+y}+\dfrac{y}{y-x}$.

14. $\dfrac{x^2+2x+4}{x+2}-\dfrac{x^2-2x+4}{2-x}$.

15. $\dfrac{1}{2a+5b}+\dfrac{3a}{25b^2-4a^2}+\dfrac{1}{2a-5b}$.

16. $\dfrac{2b-a}{x-b}-\dfrac{3x(a-b)}{b^2-x^2}+\dfrac{b-2a}{b+x}$.

17. $\dfrac{ax^2+b}{2x-1}+\dfrac{2(bx+ax^2)}{1-4x^2}-\dfrac{ax^2-b}{2x+1}$.

18. $\dfrac{a+c}{(a-b)(x-a)}+\dfrac{b+c}{(b-a)(x-b)}$.

19. $\dfrac{a-c}{(a-b)(x-a)}-\dfrac{b-c}{(b-a)(b-x)}$.

20. $\dfrac{2a+y}{(x-a)(a-b)}+\dfrac{a+b+y}{(x-b)(b-a)}-\dfrac{x+y-a}{(x-a)(x-b)}$.

21. $\dfrac{1}{(a^2-b^2)(x^2+b^2)}+\dfrac{1}{(b^2-a^2)(x^2+a^2)}-\dfrac{1}{(x^2+a^2)(x^2+b^2)}$.

22. $\dfrac{1}{x+a}+\dfrac{4a}{x^2-a^2}+\dfrac{1}{a-x}-\dfrac{2a}{x^2+a^2}$.

23. $\dfrac{3}{x+a}-\dfrac{1}{x+3a}+\dfrac{3}{a-x}+\dfrac{1}{x-3a}$.

24. $\dfrac{1}{4a^3(a+x)}-\dfrac{1}{4a^3(x-a)}+\dfrac{1}{2a^2(a^2+x^2)}-\dfrac{a^4}{a^8-x^8}$.

25. $\dfrac{x}{x^2-y^2}-\dfrac{y}{x^2+y^2}+\dfrac{x^3+y^3}{y^4-x^4}+\dfrac{xy}{(x+y)(x^2+y^2)}$.

26. $\dfrac{b}{a(a^2-b^2)}+\dfrac{a}{b(a^2+b^2)}+\dfrac{a^4+b^4}{ab(b^4-a^4)}-\dfrac{a^6}{b^8-a^8}$.

150. From Art. 149 it follows that:

(1) Changing the signs of an **odd number of factors** of numerator or denominator changes the sign before the fraction.

(2) Changing the signs of an **even number of factors** of numerator or denominator does not change the sign before the fraction.

Consider the expression

$$\frac{1}{(a-b)(a-c)}+\frac{1}{(b-c)(b-a)}+\frac{1}{(c-a)(c-b)}.$$

By changing the sign of the second factor of each denominator, we obtain

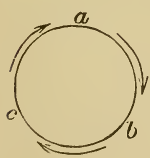
$$-\frac{1}{(a-b)(c-a)}-\frac{1}{(b-c)(a-b)}-\frac{1}{(c-a)(b-c)} \quad . \ (1).$$

Now it is readily seen that the L. C. M. of the denominators is $(a-b)(b-c)(c-a)$, and the expression

$$=\frac{-(b-c)-(c-a)-(a-b)}{(a-b)(b-c)(c-a)}$$

$$=\frac{-b+c-c+a-a+b}{(a-b)(b-c)(c-a)}=0.$$

151. There is a peculiarity in the arrangement of this example which it is desirable to notice. In the expression (1) the letters occur in what is known as **Cyclic Order**; that is, b follows a, a follows c, c follows b. Thus, if a, b, c are arranged round the circumference of a circle, as in the annexed diagram, if we start from any letter and move round in the direction of the arrows, the other letters follow in cyclic order, namely abc, bca, cab.



The observance of this principle is especially important in a large class of examples in which the differences of three letters are involved. Thus we are observing cyclic order when we write $b-c$, $c-a$, $a-b$; whereas we are violating cyclic order by the use of arrangements such as $b-c$, $a-c$, $a-b$, or $a-c$, $b-a$, $b-c$. It will always be found that the work is rendered shorter and easier by following cyclic order from the beginning, and adhering to it throughout the question.

EXAMPLES XIII. h.

Find the value of

1. $\dfrac{a}{(a-b)(a-c)}+\dfrac{b}{(b-c)(b-a)}+\dfrac{c}{(c-a)(c-b)}$.

2. $\dfrac{b}{(a-b)(a-c)}+\dfrac{c}{(b-c)(b-a)}+\dfrac{a}{(c-a)(c-b)}$.

3. $\dfrac{z}{(x-y)(x-z)}+\dfrac{x}{(y-z)(y-x)}+\dfrac{y}{(z-x)(z-y)}$.

4. $\dfrac{y+z}{(x-y)(x-z)}+\dfrac{z+x}{(y-z)(y-x)}+\dfrac{x+y}{(z-x)(z-y)}$.

5. $\dfrac{b-c}{(a-b)(a-c)}+\dfrac{c-a}{(b-c)(b-a)}+\dfrac{a-b}{(c-a)(c-b)}$.

6. $\dfrac{x^2yz}{(x-y)(x-z)}+\dfrac{y^2zx}{(y-z)(y-x)}+\dfrac{z^2xy}{(z-x)(z-y)}$.

7. $\dfrac{1+a}{(a-b)(a-c)}+\dfrac{1+b}{(b-c)(b-a)}+\dfrac{1+c}{(c-a)(c-b)}$.

8. $\dfrac{p-a}{(p-q)(p-r)}+\dfrac{q-a}{(q-r)(q-p)}+\dfrac{r-a}{(r-p)(r-q)}$.

9. $\dfrac{p+q-r}{(p-q)(p-r)}+\dfrac{q+r-p}{(q-r)(q-p)}+\dfrac{r+p-q}{(r-p)(r-q)}$.

10. $\dfrac{a^2}{(a^2-b^2)(a^2-c^2)}+\dfrac{b^2}{(b^2-c^2)(b^2-a^2)}+\dfrac{c^2}{(c^2-a^2)(c^2-b^2)}$.

11. $\dfrac{x+y}{(p-q)(p-r)}+\dfrac{x+y}{(q-r)(q-p)}+\dfrac{x+y}{(r-p)(r-q)}$.

12. $\dfrac{q+r}{(x-y)(x-z)}+\dfrac{r+p}{(y-z)(y-x)}+\dfrac{p+q}{(z-x)(z-y)}$.

CHAPTER XIV.

Complex Fractions. Mixed Expressions.

152. We now propose to consider some miscellaneous questions involving fractions of a more complicated kind than those already discussed.

In the previous chapter, the numerator and denominator have been regarded as integers; but cases frequently occur in which the numerator or denominator of a fraction is itself fractional.

153. A Complex Fraction is one that has a fraction in the numerator, or in the denominator, or in both.

Thus $\dfrac{a}{\dfrac{b}{c}}, \dfrac{\dfrac{a}{b}}{x}, \dfrac{\dfrac{a}{b}}{\dfrac{c}{d}}$ are Complex Fractions.

In the last of these types, the outside quantities, a and d, are sometimes referred to as the *extremes*, while the two middle quantities, b and c, are called the *means*.

154. By definition (Art. 148) $\dfrac{\dfrac{a}{b}}{\dfrac{c}{d}}$ is the quotient resulting from the division of $\dfrac{a}{b}$ by $\dfrac{c}{d}$; and this by Art. 143 is $\dfrac{ad}{bc}$.

$$\therefore \quad \frac{\dfrac{a}{b}}{\dfrac{c}{d}} = \frac{ad}{bc}.$$

155. From the preceding article we deduce an easy method of writing down the simplified form of a complex fraction.

Multiply the extremes for a new numerator, and the means for a new denominator.

Ex. $$\frac{\dfrac{a+x}{b}}{\dfrac{a^2-x^2}{ab}}=\frac{ab(a+x)}{b(a^2-x^2)}=\frac{a}{a-x},$$

by cancelling common factors in numerator and denominator.

156. The student should especially notice the following cases, and should be able to write off the results readily.

$$\frac{1}{\dfrac{a}{b}}=1\div\frac{a}{b}=1\times\frac{b}{a}=\frac{b}{a},$$

$$\frac{a}{\dfrac{1}{b}}=a\div\frac{1}{b}=a\times b=ab.$$

$$\frac{\dfrac{1}{a}}{\dfrac{1}{b}}=\frac{1}{a}\div\frac{1}{b}=\frac{1}{a}\times\frac{b}{1}=\frac{b}{a}.$$

157. We now proceed to show how complex fractions can be reduced by the rules already given.

Ex. 1. $$\frac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{a}{b}-\dfrac{c}{d}}=\left(\frac{a}{b}+\frac{c}{d}\right)\div\left(\frac{a}{b}-\frac{c}{d}\right)=\frac{ad+bc}{bd}\div\frac{ad-bc}{bd}$$

$$=\frac{ad+bc}{bd}\times\frac{bd}{ad-bc}=\frac{ad+bc}{ad-bc}.$$

Ex. 2. $$\frac{x+\dfrac{a^2}{x}}{x-\dfrac{a^4}{x^3}}=\left(x+\frac{a^2}{x}\right)\div\left(x-\frac{a^4}{x^3}\right)=\frac{x^2+a^2}{x}\div\frac{x^4-a^4}{x^3}$$

$$=\frac{x^2+a^2}{x}\times\frac{x^3}{x^4-a^4}=\frac{x^2}{x^2-a^2}.$$

Ex. 3. Simplify
$$\frac{\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}.$$

The numerator $=\dfrac{(a^2+b^2)^2-(a^2-b^2)^2}{(a^2+b^2)(a^2-b^2)}=\dfrac{4a^2b^2}{(a^2+b^2)(a^2-b^2)}.$

Similarly, the denominator $=\dfrac{4ab}{(a+b)(a-b)}.$

Hence the fraction $=\dfrac{4a^2b^2}{(a^2+b^2)(a^2-b^2)}\div\dfrac{4ab}{(a+b)(a-b)}$

$$=\frac{4a^2b^2}{(a^2+b^2)(a^2-b^2)}\times\frac{(a+b)(a-b)}{4ab}=\frac{ab}{a^2+b^2}.$$

Note. To ensure accuracy and neatness, when the numerator and denominator are somewhat complicated, the beginner is advised to simplify each separately as in the above example.

In the case of complex fractions like the following, called **Continued Fractions**, we begin from the lowest fraction, and simplify step by step.

Ex. 4. Simplify
$$\frac{9x^2-64}{x-1-\dfrac{1}{1-\dfrac{x}{4+x}}}.$$

The expression $=\dfrac{9x^2-64}{x-1-\dfrac{1}{\dfrac{4+x-x}{4+x}}}=\dfrac{9x^2-64}{x-1-\dfrac{4+x}{4}}$

$$=\frac{9x^2-64}{\dfrac{4x-4-(4+x)}{4}}=\frac{9x^2-64}{\dfrac{3x-8}{4}}$$

$$=\frac{4(9x^2-64)}{3x-8}=4(3x+8).$$

EXAMPLES XIV. a.

Find the value of

1. $\dfrac{\dfrac{m}{n}-\dfrac{l}{m}}{\dfrac{a}{m}-\dfrac{b}{n}}.$ 2. $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x}-\dfrac{1}{y}}.$ 3. $\dfrac{a+\dfrac{b}{d}}{x-\dfrac{y}{d}}.$ 4. $\dfrac{1+\dfrac{c}{x}}{\dfrac{b}{x}-1}.$

5. $\dfrac{2+\dfrac{3a}{4b}}{a+\dfrac{8b}{3}}$.

7. $\dfrac{1-\dfrac{y^2}{x^2}}{1+\dfrac{y^2}{x^2}}$.

9. $\dfrac{a}{b+\dfrac{c}{d}}$.

11. $\dfrac{\dfrac{a}{b}+\dfrac{c}{d}}{\dfrac{m}{n}+\dfrac{k}{p}}$.

6. $\dfrac{3a+\dfrac{7b}{8c}}{3c+\dfrac{7b}{8a}}$.

8. $\dfrac{1}{a+\dfrac{b}{c}}$.

10. $\dfrac{x}{x-\dfrac{m}{n}}$.

12. $\dfrac{x-\dfrac{1}{x}}{1+\dfrac{1}{x}}$.

13. $\dfrac{x+5+\dfrac{6}{x}}{1+\dfrac{6}{x}+\dfrac{8}{x^2}}$.

14. $\dfrac{\dfrac{1}{x}-\dfrac{2}{x^2}-\dfrac{3}{x^3}}{\dfrac{9}{x}-x}$.

15. $\dfrac{2x^2-x-6}{\dfrac{4}{x^2}-1}$.

16. $\dfrac{2}{1-x^2}\div\left(\dfrac{1}{1-x}-\dfrac{1}{1+x}\right)$.

17. $\left(\dfrac{a^3-b^3}{a-b}-\dfrac{a^3+b^3}{a+b}\right)\div\dfrac{4ab}{a^2-b^2}$.

18. $\left(\dfrac{a^2-ax+x^2}{a-x}-\dfrac{a^2+ax+x^2}{a+x}\right)\div\dfrac{x^3}{a^2-x^2}$.

19. $\left(y+\dfrac{xy}{y-x}\right)\left(y-\dfrac{xy}{x+y}\right)\times\dfrac{y^2-x^2}{y^2+x^2}$.

20. $\left(\dfrac{x}{1+x}+\dfrac{1-x}{x}\right)\div\left(\dfrac{x}{1+x}-\dfrac{1-x}{x}\right)$.

21. $\dfrac{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}{1-\dfrac{a^2+b^2}{(a+b)^2}}$.

26. $\dfrac{1}{4x+\dfrac{4x}{1+\dfrac{2(x+y)}{6-x}}}$.

22. $\dfrac{\dfrac{a}{x^2}+\dfrac{x}{a^2}}{\dfrac{1}{a^2}-\dfrac{1}{ax}+\dfrac{1}{x^2}}$.

27. $\dfrac{a}{x+\dfrac{m}{y+\dfrac{n}{z}}}$.

23. $\dfrac{\dfrac{1}{3x-2}-\dfrac{1}{3x+2}}{\dfrac{1}{9-\dfrac{4}{x^2}}}$.

28. $\dfrac{1}{1-\dfrac{1+x}{x-\dfrac{1}{x}}}$.

24. $1+\dfrac{x}{1+x+\dfrac{2x^2}{1-x}}$.

29. $\dfrac{x-2}{x-2-\dfrac{x}{x-\dfrac{x-1}{x-2}}}$.

25. $\dfrac{1}{a-\dfrac{a^2-1}{a+\dfrac{1}{a-1}}}$.

30. $\dfrac{1}{x-\dfrac{1}{x+\dfrac{1}{x}}}-\dfrac{1}{x+\dfrac{1}{x-\dfrac{1}{x}}}$.

158. Sometimes it is convenient to express a single fraction as a group of fractions.

Ex. $$\frac{5x^2y - 10xy^2 + 15y^3}{10x^2y^2} = \frac{5x^2y}{10x^2y^2} - \frac{10xy^2}{10x^2y^2} + \frac{15y^3}{10x^2y^2}$$
$$= \frac{1}{2y} - \frac{1}{x} + \frac{3y}{2x^2}.$$

MIXED EXPRESSIONS.

159. We may often express a fraction in an equivalent form, partly integral and partly fractional. It is then called a **Mixed Expression.**

Ex. 1. $$\frac{x+7}{x+2} = \frac{(x+2)+5}{x+2} = 1 + \frac{5}{x+2}.$$

Ex. 2. $$\frac{3x-2}{x+5} = \frac{3(x+5)-15-2}{x+5} = \frac{3(x+5)-17}{x+5} = 3 - \frac{17}{x+5}.$$

In some cases actual division may be advisable.

Ex. 3. Show that $\dfrac{2x^2 - 7x - 1}{x-3} = 2x - 1 - \dfrac{4}{x-3}.$

Performing the indicated division, we obtain a quotient $2x - 1$, and a remainder -4.

Therefore $$\frac{2x^2 - 7x - 1}{x-3} = 2x - 1 - \frac{4}{x-3}.$$

160. If the numerator be of lower dimensions (Art. 29) than the denominator, we may still perform the division, and express the result in a form which is partly integral and partly fractional.

Ex. Prove that $\dfrac{2x}{1+3x^2} = 2x - 6x^3 + 18x^5 - \dfrac{54x^7}{1+3x^2}.$

By division
$$\begin{array}{rl}
1+3x^2)\,2x & (2x - 6x^3 + 18x^5 \\
\underline{2x + 6x^3} & \\
-6x^3 & \\
\underline{-6x^3 - 18x^5} & \\
18x^5 & \\
\underline{18x^5 + 54x^7} & \\
-54x^7 &
\end{array}$$

whence the result follows.

Here the division may be carried on to any number of terms in the quotient, and we can stop at any term we please by taking for our remainder the fraction whose numerator is the remainder last found, and whose denominator is the divisor.

Thus, if we carried on the quotient to four terms, we should have

$$\frac{2x}{1+3x^2}=2x-6x^3+18x^5-54x^7+\frac{162x^9}{1+3x^2}.$$

The terms in the quotient may be fractional; thus if x^2 is divided by x^3-a^3, the first four terms of the quotient are $\frac{1}{x}+\frac{a^3}{x^4}+\frac{a^6}{x^7}+\frac{a^9}{x^{10}}$, and the remainder is $\frac{a^{12}}{x^{10}}$.

161. Miscellaneous examples in multiplication and division occur which can be dealt with by the preceding rules for the reduction of fractions.

Ex. Multiply $x+2a-\frac{a^2}{2x+3a}$ by $2x-a-\frac{2a^2}{x+a}$.

$$\text{The product}=\left(x+2a-\frac{a^2}{2x+3a}\right)\times\left(2x-a-\frac{2a^2}{x+a}\right)$$

$$=\frac{2x^2+7ax+6a^2-a^2}{2x+3a}\times\frac{2x^2+ax-a^2-2a^2}{x+a}$$

$$=\frac{2x^2+7ax+5a^2}{2x+3a}\times\frac{2x^2+ax-3a^2}{x+a}$$

$$=\frac{(2x+5a)(x+a)}{2x+3a}\times\frac{(2x+3a)(x-a)}{x+a}$$

$$=(2x+5a)(x-a).$$

EXAMPLES XIV. b.

Express each of the following fractions as a group of simple fractions in lowest terms:

1. $\frac{3x^2y+xy^2-y^3}{9xy}$.
2. $\frac{3a^3x-4a^2x^2+6ax^3}{12ax}$.
3. $\frac{a^3-3a^2b+3ab^2+b^3}{2ab}$.
4. $\frac{a+b+c}{abc}$.
5. $\frac{bc+ca+ab}{abc}$.
6. $\frac{a^3bc-3ab^3c+2abc}{6abc}$.

Perform the following divisions, giving the remainder after four terms in the quotient:

7. $x \div (1+x)$. 9. $(1+x)\div(1-x)$. 11. $x^2 \div (x+3)$.

8. $a \div (a-b)$. 10. $1 \div (1-x+x^2)$. 12. $1 \div (1-x)^2$.

13. Show that $\dfrac{a^3-b^3}{(a-b)^2} = a+2b+\dfrac{3b^2}{a-b}$.

14. Show that $x^2-xy+y^2-\dfrac{2y^3}{x+y}=\dfrac{x^3-y^3}{x+y}$.

15. Show that $\dfrac{60x^3-17x^2-4x+1}{5x^2+9x-2}=12x-25+\dfrac{49}{x+2}$.

16. Show that $1+\dfrac{a^2+b^2-c^2}{2ab}=\dfrac{(a+b+c)(a+b-c)}{2ab}$.

17. Divide $x+\dfrac{16x-27}{x^2-16}$ by $x-1+\dfrac{13}{x+4}$.

18. Multiply $a^2-2ax+4x^2-\dfrac{16x^3}{a+2x}$ by $3-\dfrac{6x(a+4x)}{a^2+2ax+4x^2}$.

19. Divide $b^2+3b-2-\dfrac{12}{b-3}$ by $3b+6-\dfrac{2b^2}{b-3}$.

20. Divide $a^2+9b^2+\dfrac{65b^4}{a^2-9b^2}$ by $a+3b+\dfrac{13b^2}{a-3b}$.

21. Multiply $4x^2+14x+\dfrac{98x-27}{2x-7}$ by $\dfrac{1}{6}-\dfrac{3x+29}{12x^2+18x+27}$.

162. We add an exercise in which most of the processes connected with fractions will be illustrated.

EXAMPLES XIV. c.

Simplify the following fractions:

1. $\dfrac{4a(a^2-x^2)}{3b(c^2-x^2)} \div \left[\dfrac{a^2-ax}{bc+bx}\times\dfrac{a^2+2ax+x^2}{c^2-2cx+x^2}\right]$.

2. $\dfrac{x(x+a)(x+2a)}{3a}-\dfrac{x(x+a)(2x+a)}{6a}$.

3. $\dfrac{1}{b}\left(\dfrac{1}{a-b}-\dfrac{1}{a+2b}\right)-\dfrac{2}{a^2+ab-2b^2}$. 4. $\left(\dfrac{x+y}{x-y}\right)^2-\left(\dfrac{x-y}{x+y}\right)^2$.

5. $\dfrac{2}{x-1}+\dfrac{2}{x+1}-\dfrac{4x}{x^2-x+1}$.

6. $\left(\dfrac{x^2}{1-x^4}+\dfrac{2x^4}{1-x^8}\right)\div\left(\dfrac{x^2+1}{x}\right)^2$

7. $\dfrac{1}{x}-\dfrac{1}{(x+1)^2}-\dfrac{2}{x+1}+\dfrac{x}{1+x+x^2}.$

8. $\dfrac{1+x^3}{1+2x+2x^2+x^3}.$

9. $\dfrac{2x^3-9x^2+27}{3x^3-81x+162}.$

10. $\dfrac{a}{b}-\dfrac{(a^2-b^2)x}{b^2}+\dfrac{a(a^2-b^2)x^2}{b^2(b+ax)}.$

11. $\left\{\dfrac{x^4-a^4}{x^2-2ax+a^2}\div\dfrac{x^2+ax}{x-a}\right\}\times\dfrac{x^5-a^2x^3}{x^3+a^3}\div\left(\dfrac{x}{a}-\dfrac{a}{x}\right).$

12. $\dfrac{a^2-x^2}{a^2+ax+x^2}\div\dfrac{\left(1-\dfrac{x}{a}\right)^3\left(1+\dfrac{x}{a}\right)^3}{a^3-x^3}.$

13. $\dfrac{x^4-2x^2+1}{3x^5-10x^3+15x-8}.$

14. $\dfrac{a^3+a(1+a)y+y^2}{a^4-y^2}.$

15. $\dfrac{1}{a}+\dfrac{2}{a+1}+\dfrac{3}{a+2}-\dfrac{\dfrac{4}{a}}{1+\dfrac{1}{a}}.$

16. $\dfrac{x+3}{2x^2+9x+9}+\dfrac{1}{2}\times\dfrac{1}{2x-3}-\dfrac{1}{x-\dfrac{9}{4x}}.$

17. $\dfrac{2}{x^3+x^2+x+1}-\dfrac{2}{x^3-x^2+x-1}.$

18. $\dfrac{1-a^2}{(1+ax)^2-(a+x)^2}\div\dfrac{1}{2}\left(\dfrac{1}{1-x}+\dfrac{1}{1+x}\right).$

19. $\dfrac{2x^3-x^2-2x+1}{x^3-3x+2}.$

20. $\dfrac{x^2-6x+8}{4x^3-21x^2+15x+20}.$

21. $\dfrac{2a}{(x-2a)^2}-\dfrac{x-a}{x^2-5ax+6a^2}+\dfrac{2}{x-3a}.$

22. $\dfrac{1}{2}\left(\dfrac{a^2+x^2}{a^2-x^2}\right)-\dfrac{1}{2}\times\dfrac{a+x}{a-x}-\left(\dfrac{a}{a+x}\right)^2.$

23. $\dfrac{x}{2}\left(\dfrac{1}{x-y}-\dfrac{1}{x+y}\right)\times\dfrac{x^2-y^2}{x^2y+xy^2}\div\dfrac{1}{x+y}.$

24. $\dfrac{1}{x+y}\div\left[\dfrac{y}{2}\left(\dfrac{1}{x+y}+\dfrac{1}{x-y}\right)\times\dfrac{x^2-y^2}{x^2y+xy^2}\right].$

25. $\left(3x-5-\dfrac{2}{x}\right)\left(3x+5-\dfrac{2}{x}\right)\div\left(x-\dfrac{4}{x}\right).$

26. $\left\{\dfrac{2}{x}-\dfrac{1}{a+x}+\dfrac{1}{a-x}\right\}\div\left(\dfrac{a+x}{a-x}-\dfrac{a-x}{a+x}\right).$

27. $\dfrac{1}{2x-1}-\dfrac{2x-\dfrac{1}{2x}}{4x^2-1}.$

28. $\left(b+\dfrac{ab}{b-a}\right)\left(b-\dfrac{ab}{a+b}\right)\left(\dfrac{b^2-a^2}{b^2+a^2}\right).$

29. $\left\{\dfrac{b+\dfrac{a-b}{1+ab}}{1-\dfrac{(a-b)b}{1+ab}}-\dfrac{a-\dfrac{a-b}{1-ab}}{1-\dfrac{a(a-b)}{1-ab}}\right\}\div\left(\dfrac{a}{b}-\dfrac{b}{a}\right).$

30. $\dfrac{\dfrac{x^2+y^2}{y}-x}{\dfrac{1}{y}-\dfrac{1}{x}}\times\dfrac{x^2-y^2}{x^3+y^3}.$

31. $\dfrac{2(x^2-\frac{1}{4})}{2x+1}+\frac{1}{2}.$

32. $\dfrac{a+\dfrac{b-a}{1+ab}}{1-\dfrac{a(b-a)}{1+ab}}\times\dfrac{\dfrac{x+y}{1-xy}-y}{1+\dfrac{y(x+y)}{1-xy}}.$

33. $\dfrac{\dfrac{a+b}{a-b}+\dfrac{a-b}{a+b}}{\dfrac{a-b}{a+b}-\dfrac{a+b}{a-b}}\times\dfrac{ab^3-a^3b}{a^2+b^2}.$

34. $\dfrac{(1-x^2)(1-x^3)}{x(1+x)(1-x)^2}-\dfrac{x^3+\dfrac{1}{x^3}}{x^2+\dfrac{1}{x^2}-1}.$

35. $\left\{x^3-\dfrac{1}{x^3}-3\left(x-\dfrac{1}{x}\right)\right\}\div\left(x-\dfrac{1}{x}\right).$

36. $\dfrac{1+\dfrac{1}{m}}{\dfrac{1}{m}}\times\dfrac{\dfrac{1}{m}}{m^2+\dfrac{1}{m}}\div\dfrac{\dfrac{1}{m}}{m-1+\dfrac{1}{m}}.$

37. $\dfrac{\dfrac{x}{y}+\dfrac{y}{x}-1}{\dfrac{x^2}{y^2}+\dfrac{x}{y}+1}\times\dfrac{1+\dfrac{y}{x}}{x-y}\div\dfrac{1+\dfrac{y^3}{x^3}}{\dfrac{x^2}{y}-\dfrac{y^2}{x}}.$

38. $\dfrac{1}{a^2-2}-\dfrac{2}{a^2-1}+\dfrac{2}{a^2+1}-\dfrac{1}{a^2+2}.$

39. $\dfrac{1}{6m-2n}+\dfrac{1}{3m+2n}-\dfrac{3}{6m+2n}.$

40. $\dfrac{3}{4(1-x)^2}+\dfrac{3}{8(1-x)}+\dfrac{1}{8(1+x)}+\dfrac{x-1}{4(1+x^2)}.$

41. $\dfrac{4}{9(x-2)}+\dfrac{5}{9(x+1)}-\dfrac{1}{3(x+1)^2}-\dfrac{1}{x+2+\dfrac{1}{x}}.$

42. $\left(\frac{x^2}{y}+\frac{y^2}{x}\right)\left(\frac{1}{y^2-x^2}\right)-\frac{y}{x^2+xy}+\frac{x}{xy-y^2}.$

43. $\frac{x^2-(y-z)^2}{(x+z)^2-y^2}+\frac{y^2-(z-x)^2}{(x+y)^2-z^2}+\frac{z^2-(x-y)^2}{(y+z)^2-x^2}.$

44. $\frac{x^2-(y-2z)^2}{(2z+x)^2-y^2}+\frac{y^2-(2z-x)^2}{(x+y)^2-4z^2}+\frac{4z^2-(x-y)^2}{(y+2z)^2-x^2}.$

45. $\frac{(x-y)(y-z)+(y-z)(z-x)+(z-x)(x-y)}{x(z-x)+y(x-y)+z(y-z)}.$

46. $\frac{a-b-c}{(a-b)(a-c)}+\frac{b-c-a}{(b-c)(b-a)}+\frac{c-a-b}{(c-a)(c-b)}.$

47. $\frac{c+a}{(a-b)(a-c)}+\frac{a+b}{(b-c)(b-a)}+\frac{b+c}{(c-a)(c-b)}.$

48. $\frac{x^2-(2y-3z)^2}{(3z+x)^2-4y^2}+\frac{4y^2-(3z-x)^2}{(x+2y)^2-9z^2}+\frac{9z^2-(x-2y)^2}{(2y+3z)^2-x^2}.$

49. $\frac{9y^2-(4z-2x)^2}{(2x+3y)^2-16z^2}+\frac{16z^2-(2x-3y)^2}{(3y+4z)^2-4x^2}+\frac{4x^2-(3y-4z)^2}{(4z+2x)^2-9y^2}.$

50. $\dfrac{\frac{1}{x}-\frac{x+a}{x^2+a^2}}{\frac{1}{a}-\frac{a+x}{a^2+x^2}}+\dfrac{\frac{1}{x}-\frac{x-a}{x^2+a^2}}{\frac{1}{a}-\frac{a-x}{a^2+x^2}}.$

51. $\frac{(x+a)(x+b)-(y+a)(y+b)}{x-y}-\frac{(x-a)(y-b)-(x-b)(y-a)}{(a-b)}.$

52. $\left(\frac{a+x}{a^2-ax+x^2}-\frac{a-x}{a^2+ax+x^2}\right)\div\left(\frac{a^2+x^2}{a^3-x^3}-\frac{a^2-x^2}{a^3+x^3}\right).$

53. $\dfrac{\frac{x-1}{3}+\frac{x-1}{x-2}}{\frac{x+2}{4}+\frac{x+2}{x-3}}\div\dfrac{\frac{x+3}{7}-\frac{x+3}{x+4}}{\frac{x-2}{3}+\frac{x-2}{x-1}}.$

54. $\dfrac{\left(\frac{3x+x^3}{1+3x^2}\right)^2-1}{\frac{3x^2-1}{x^3-3x}+1}\div\dfrac{\frac{9}{x^2}-\frac{33-x^2}{3x^2+1}}{\frac{3}{x^2}-\frac{2(x^2+3)}{(x^3-x)^2}}.$

CHAPTER XV.

Fractional and Literal Equations.

163. In this chapter we propose to give a miscellaneous collection of equations. Some of these will serve as a useful exercise for revision of the methods already explained in previous chapters; but we also add others presenting more difficulty, the solution of which will often be facilitated by some special artifice.

The following examples worked in full will sufficiently illustrate the most useful methods.

Ex. 1. Solve $4 - \frac{x-9}{8} = \frac{x}{22} - \frac{1}{2}$.

Multiply by 88, which is the least common multiple of the denominators, and we get

$$352 - 11(x-9) = 4x - 44;$$

removing brackets, $\quad 352 - 11x + 99 = 4x - 44;$

transposing, $\quad -11x - 4x = -44 - 352 - 99;$

collecting terms and changing signs, $\quad 15x = 495;$

$$\therefore\ x = 33.$$

Note. In this equation $-\frac{x-9}{8}$ is regarded as a single term with the minus sign before it. In fact it is equivalent to $-\frac{1}{8}(x-9)$, the line between the numerator and denominator having the same effect as a bracket.

Ex. 2. Solve $\frac{8x+23}{20} - \frac{5x+2}{3x+4} = \frac{2x+3}{5} - 1$.

Multiply by 20, and we have

$$8x + 23 - \frac{20(5x+2)}{3x+4} = 8x + 12 - 20.$$

By transposition, $\quad 31 = \frac{20(5x+2)}{3x+4}.$

Multiplying across, $93x + 124 = 20(5x + 2)$,

$$84 = 7x;$$

$$\therefore\ x = 12.$$

Ex. 3. Solve $\dfrac{x-8}{x-10} + \dfrac{x-4}{x-6} = \dfrac{x-5}{x-7} + \dfrac{x-7}{x-9}$.

This equation might be solved by clearing of fractions, but the work would be very laborious. The solution will be much simplified by proceeding as follows:

Transposing, $\dfrac{x-8}{x-10} - \dfrac{x-5}{x-7} = \dfrac{x-7}{x-9} - \dfrac{x-4}{x-6}$.

Simplifying each side *separately*, we have

$$\frac{(x-8)(x-7)-(x-5)(x-10)}{(x-10)(x-7)} = \frac{(x-7)(x-6)-(x-4)(x-9)}{(x-9)(x-6)};$$

$$\therefore \frac{x^2-15x+56-(x^2-15x+50)}{(x-10)(x-7)} = \frac{x^2-13x+42-(x^2-13x+36)}{(x-9)(x-6)};$$

$$\therefore\ \frac{6}{(x-10)(x-7)} = \frac{6}{(x-9)(x-6)}.$$

Hence, since the numerators are equal, the denominators must be equal; that is,

$$(x-10)(x-7) = (x-9)(x-6),$$

$$x^2 - 17x + 70 = x^2 - 15x + 54,$$

$$16 = 2x;$$

$$\therefore\ x = 8.$$

Ex. 4. Solve $\dfrac{5x-64}{x-13} - \dfrac{2x-11}{x-6} = \dfrac{4x-55}{x-14} - \dfrac{x-6}{x-7}$.

We have $5 + \dfrac{1}{x-13} - \left(2 + \dfrac{1}{x-6}\right) = 4 + \dfrac{1}{x-14} - \left(1 + \dfrac{1}{x-7}\right)$;

$$\therefore\ \frac{1}{x-13} - \frac{1}{x-6} = \frac{1}{x-14} - \frac{1}{x-7}.$$

Simplifying each side *separately*, we have

$$\frac{7}{(x-13)(x-6)} = \frac{7}{(x-14)(x-7)},$$

$$(x-13)(x-6) = (x-14)(x-7),$$

$$x^2 - 19x + 78 = x^2 - 21x + 98,$$

$$2x = 20;$$

$$\therefore\ x = 10.$$

164. To solve equations whose coefficients are decimals, we may express the decimals as common fractions, and proceed as before; but it is often found more simple to work entirely in decimals.

Ex. 1. Solve $.375x - 1.875 = .12x + 1.185.$

Transposing, $.375x - .12x = 1.185 + 1.875;$

collecting terms, $(.375 - .12)x = 3.06,$

that is, $.255x = 3.06;$

$$\therefore\ x = \frac{3.06}{.255} = 12.$$

Ex. 2. Solve $.\dot{6}x + .25 - \frac{1}{9}x = 1.\dot{8} - .75x - \frac{1}{3}.$

Expressing the decimals as common fractions, we have

$$\tfrac{2}{3}x + \tfrac{1}{4} - \tfrac{1}{9}x = 1\tfrac{8}{9} - \tfrac{3}{4}x - \tfrac{1}{3};$$

clearing of fractions, $24x + 9 - 4x = 68 - 27x - 12;$

transposing, $24x - 4x + 27x = 68 - 12 - 9,$

$$47x = 47;$$

$$\therefore\ x = 1.$$

EXAMPLES XV. a.

1. $\dfrac{4(x+2)}{5} = 7 + \dfrac{5x}{13}.$

2. $\dfrac{x+4}{14} + \dfrac{x-4}{6} = 2.$

3. $\dfrac{x+20}{9} + \dfrac{3x}{7} = 6.$

4. $\dfrac{x-8}{7} + \dfrac{x-3}{3} + \dfrac{5}{21} = 0.$

5. $\dfrac{5(x+5)}{8} - \dfrac{2(x-3)}{7} = 5\tfrac{19}{28}.$

6. $\dfrac{x}{2} + \dfrac{x}{3} - \dfrac{x}{4} + \dfrac{x}{5} = 7\tfrac{5}{6}.$

7. $\dfrac{3x}{4} - \dfrac{6}{17}(x+10) - (x-3) = \dfrac{x-7}{51} - 4\tfrac{3}{4}.$

8. $3 + \dfrac{x}{4} = \dfrac{1}{2}\left(4 - \dfrac{x}{3}\right) - \dfrac{5}{6} + \dfrac{1}{3}\left(11 - \dfrac{x}{2}\right).$

9. $\dfrac{1}{5}(x-8) + \dfrac{x+4}{4} + \dfrac{x-1}{7} = 7 - \dfrac{23-x}{5}.$

10. $x - \left(3x - \dfrac{2x-5}{10}\right) = \dfrac{1}{6}(2x-57) - \dfrac{5}{3}.$

11. $\dfrac{2x-5}{5} + \dfrac{x-3}{2x-15} = \dfrac{4x-3}{10} - 1\tfrac{1}{10}.$

12. $\dfrac{4(x+3)}{9} = \dfrac{8x+37}{18} - \dfrac{7x-29}{5x-12}.$

13. $\dfrac{(2x-1)(3x+8)}{6x(x+4)} - 1 = 0.$

14. $\dfrac{2x+5}{5x+3} - \dfrac{2x+1}{5x+2} = 0.$

15. $\dfrac{4}{x+3} - \dfrac{2}{x+1} = \dfrac{5}{2x+6} - \dfrac{2\frac{1}{2}}{2x+2}.$

16. $\dfrac{7}{x-4} - \dfrac{60}{5x-30} = \dfrac{10\frac{1}{2}}{3x-12} - \dfrac{8}{x-6}.$

17. $\dfrac{3}{4-2x} + \dfrac{30}{8(1-x)} = \dfrac{3}{2-x} + \dfrac{5}{2-2x}.$

18. $\dfrac{25-\dfrac{x}{3}}{x+1} + \dfrac{16x+4\frac{1}{5}}{3x+2} = 5 + \dfrac{23}{x+1}.$

19. $\dfrac{30+6x}{x+1} + \dfrac{60+8x}{x+3} = 14 + \dfrac{48}{x+1}.$

20. $\dfrac{x}{x-2} - \dfrac{x+1}{x-1} = \dfrac{x-8}{x-6} - \dfrac{x-9}{x-7}.$

21. $\dfrac{x+5}{x+4} - \dfrac{x-6}{x-7} = \dfrac{x-4}{x-5} - \dfrac{x-15}{x-16}.$

22. $\dfrac{x-7}{x-9} - \dfrac{x-9}{x-11} = \dfrac{x-13}{x-15} - \dfrac{x-15}{x-17}.$

23. $\dfrac{x+3}{x+6} - \dfrac{x+6}{x+9} = \dfrac{x+2}{x+5} - \dfrac{x+5}{x+8}.$

24. $\dfrac{x+2}{x} + \dfrac{x-7}{x-5} - \dfrac{x+3}{x+1} = \dfrac{x-6}{x-4}.$

25. $\dfrac{4x-17}{x-4} + \dfrac{10x-13}{2x-3} = \dfrac{8x-30}{2x-7} + \dfrac{5x-4}{x-1}.$

26. $\dfrac{5x-8}{x-2} + \dfrac{6x-44}{x-7} - \dfrac{10x-8}{x-1} = \dfrac{x-8}{x-6}.$

27. $\dfrac{2x-3}{.3x-.4} = \dfrac{.4x-.6}{.06x-.07}.$

28. $\dfrac{x-2}{.0\dot{5}} - \dfrac{x-4}{.0625} = 56.$

29. $.08\dot{3}(x-.625) = .\dot{0}\dot{9}(x-.59375).$

30. $(2x+1.5)(3x-2.25) = (2x-1.125)(3x+1.25).$

31. $\dfrac{.3x-1}{.5x-.4} = \dfrac{.5+1.2x}{2x-.1}.$

32. $\dfrac{1-1.4x}{.2+x} = \dfrac{.7(x-1)}{.1-.5x}.$

33. $\dfrac{(.3x-2)(.3x-1)}{.2x-1} - \dfrac{1}{6}(.3x-2) = .4x-2.$

LITERAL EQUATIONS.

165. In the equations we have discussed hitherto the coefficients have been numerical quantities, but equations often involve *literal* coefficients. [Art. 6.] These are supposed to be known, and will appear in the solution.

Ex. 1. Solve $(x+a)(x+b) - c(a+c) = (x-c)(x+c) + ab.$

Multiplying out, we have

$$x^2 + ax + bx + ab - ac - c^2 = x^2 - c^2 + ab;$$

whence

$$ax + bx = ac,$$

$$(a+b)x = ac;$$

$$\therefore x = \frac{ac}{a+b}.$$

Ex. 2. Solve $\dfrac{a}{x-a}-\dfrac{b}{x-b}=\dfrac{a-b}{x-c}$.

Simplifying the left side, we have

$$\frac{a(x-b)-b(x-a)}{(x-a)(x-b)}=\frac{a-b}{x-c},$$

$$\frac{(a-b)x}{(x-a)(x-b)}=\frac{a-b}{x-c};$$

$$\therefore \frac{x}{(x-a)(x-b)}=\frac{1}{x-c}.$$

Multiplying across, $x^2-cx=x^2-ax-bx+ab,$

$$ax+bx-cx=ab,$$

$$(a+b-c)x=ab;$$

$$\therefore x=\frac{ab}{a+b-c}.$$

EXAMPLES XV. b.

1. $ax-2b=5bx-3a.$
2. $a^2(x-a)+b^2(x-b)=abx.$
3. $x^2+a^2=(b-x)^2.$
4. $(x-a)(x+b)=(x-a+b)^2.$
5. $a(x-2)+2x=6+a.$
6. $m^2(m-x)-mnx=n^2(n+x).$
7. $(a+x)(b+x)=x(x-c).$
8. $(a-b)(x-a)=(a-c)(x-b).$
9. $\dfrac{2x+3a}{x+a}=\dfrac{2(3x+2a)}{3x+a}.$
10. $\dfrac{2(x-b)}{3x-c}=\dfrac{2x+b}{3(x-c)}.$
11. $\dfrac{1}{a}-\dfrac{1}{x}=\dfrac{1}{x}-\dfrac{1}{b}.$
12. $\dfrac{2}{3}\left(\dfrac{x}{a}+1\right)=\dfrac{3}{4}\left(\dfrac{x}{a}-1\right).$
13. $\dfrac{a}{x}=c(a-b)+\dfrac{b}{x}.$
14. $\dfrac{9a}{b}-\dfrac{3x}{b}=\dfrac{4b}{a}-\dfrac{2x}{a}.$
15. $\dfrac{x-a}{b-x}=\dfrac{x-b}{a-x}.$
16. $\dfrac{x-a}{2}=\dfrac{(x-b)^2}{2x-a}.$
17. $\dfrac{1}{4}x(x-a)-\left(\dfrac{x+a}{2}\right)^2=\dfrac{2a}{3}\left(x-\dfrac{a}{2}\right).$
18. $(a+b)x^2-a(bx+a^2)=bx(x-a)+ax(x-b).$
19. $b(a+x)-(a+x)(b-x)=x^2+\dfrac{bc^2}{a}.$
20. $b(a-x)-\dfrac{a}{b}(b+x)^2+ab\left(\dfrac{x}{b}+1\right)^2=0.$
21. $x^2+a(2a-x)-\dfrac{3b^2}{4}=\left(x-\dfrac{b}{2}\right)^2+a^2.$
22. $(2x-a)\left(x+\dfrac{2a}{3}\right)=4x\left(\dfrac{a}{3}-x\right)-\dfrac{1}{2}(a-4x)(2a+3x).$

CHAPTER XVI.

Problems leading to Fractional and Literal Equations.

166. We here give some problems which lead to equations with fractional and literal coefficients.

Ex. 1. Find two numbers which differ by 4, and such that one-half of the greater exceeds one-sixth of the less by 8.

Let x represent the smaller number, then $x + 4$ represents the greater.

One-half of the greater is represented by $\frac{1}{2}(x+4)$, and one-sixth of the less by $\frac{1}{6}x$.

Hence $\frac{1}{2}(x+4) - \frac{1}{6}x = 8;$

multiplying by 6, $3x + 12 - x = 48;$

$$2x = 36;$$

$$\therefore\ x = 18, \text{ the less number,}$$

and $x + 4 = 22$, the greater.

Ex. 2. A has \$180, and B has \$84; after B has won from A a certain sum, A has then five-sixths of what B has; how much did B win?

Suppose that B wins x dollars, A has then $180 - x$ dollars, and B has $84 + x$ dollars.

Hence $180 - x = \frac{5}{6}(84 + x);$

$$1080 - 6x = 420 + 5x,$$

$$11x = 660;$$

$$\therefore\ x = 60.$$

Therefore B wins \$60.

EXAMPLES XVI.

1. Find a number such that the sum of its sixth and ninth parts may be equal to 15.

2. What is the number whose eighth, sixth, and fourth parts together make up 13?

3. There is a number whose fifth part is less than its fourth part by 3: find it.

4. Find a number such that six-sevenths of it shall exceed four-fifths of it by 2.

5. The fifth, fifteenth, and twenty-fifth parts of a number together make up 23: find the number.

6. Two consecutive numbers are such that one-fourth of the less exceeds one-fifth of the greater by 1: find the numbers.

7. Two numbers differ by 28, and one is eight-ninths of the other: find them.

8. There are two consecutive numbers such that one-fifth of the greater exceeds one-seventh of the less by 3: find them.

9. Find three consecutive numbers such that if they be divided by 10, 17, and 26, respectively, the sum of the quotients will be 10.

10. A and B begin to play with equal sums, and when B has lost five-elevenths of what he had to begin with, A has gained \$6 more than half of what B has left: what had they at first?

11. From a certain number 3 is taken, and the remainder is divided by 4; the quotient is then increased by 4 and divided by 5, and the result is 2: find the number.

12. In a cellar one-fifth of the wine is port and one-third claret: besides this it contains 15 dozen of sherry and 30 bottles of hock: how much port and claret does it contain?

13. Two-fifths of A's money is equal to B's, and seven-ninths of B's is equal to C's, in all they have \$770: what have they each?

14. A, B, and C have \$1285 among them: A's share is greater than five-sixths of B's by \$25, and C's is four-fifteenths of B's: find the share of each.

15. A man sold a horse for \$35 and half as much as he gave for it, and gained thereby \$10: what did he pay for the horse?

16. The width of a room is two-thirds of its length. If the width had been 3 feet more, and the length 3 feet less, the room would have been square: find its dimensions.

17. What is the property of a person whose income is \$430, when he has two-thirds of it invested at 4 per cent, one-fourth at 3 per cent, and the remainder at 2 per cent?

18. I bought a certain number of apples at three for a cent, and five-sixths of that number at four for a cent: by selling them at sixteen for six cents I gain $3\frac{1}{2}$ cents: how many apples did I buy?

19. Find two numbers such that the one may be n times as great as the other, and their sum equal to b.

20. A man agreed to work a days on these conditions: for each day he worked he was to receive c cents, and for each day he was idle he was to forfeit d cents. At the end of a days he received m cents. How many days was he idle?

21. A sum of money is divided among three persons: the first receives a dollars more than a third of the whole sum; the second receives b dollars more than a half of what remains; and the third receives c dollars, the amount which is left. Find the original sum.

22. Out of a certain sum a man paid \$96; he loaned half of the remainder, and then spent one-fifth of what he had left. After these deductions he still had one-tenth of the original sum. How much had he at first?

23. A man moves 12 miles in an hour and a half, rowing with the tide, and requires 4 hours to return, rowing against a tide one-quarter as strong: find the velocity of the stronger tide.

24. A man moves a miles in b hours, rowing with the tide, but requires c hours to return, rowing against a tide d times as strong as the first: find the velocity of the stronger tide.

25. A has a certain sum of money from which he gives to B \$4 and one-sixth of what remains; he then gives to C \$5 and one-fifth of what remains, and finds that he has given away half of his money. How many dollars had A, and how many dollars did B receive?

26. The fore-wheel of a carriage is a feet, and the hind-wheel is b feet in circumference. What is the distance passed over when the fore-wheel has made c revolutions more than the hind-wheel?

27. In a certain weight of gunpowder the nitre composed 10 pounds more than two-thirds of the weight, the sulphur $4\frac{1}{2}$ pounds less than one-sixth, and the charcoal $5\frac{1}{2}$ pounds less than one-fifth of the nitre. What was the weight of the gunpowder?

28. Two-thirds of A's money is equal to B's, and three-fourths of B's is equal to C's; together they have \$650. What amount has each?

29. A dealer spends \$1450 in buying horses at \$100 each and cows at \$30 each; through disease he loses 10 per cent of the horses and 20 per cent of the cows. By selling the remainder at the price he gave for them he receives \$1260: find how many of each kind he bought.

30. A, B, C start from the same place at the rates of c, $c+d$, $c+2d$ miles an hour respectively: B starts k hours after A; how long after B must C start in order that they may overtake A at the same instant, and how far will they then have walked?

MISCELLANEOUS EXAMPLES III.

1. Subtract $p^3 - 4p^2 + 8$ from unity, and $3p^2 - p - 7$ from zero, and add the results.

2. Simplify $(x-y)^2 + (x-z)^2 + 2\{(x-y)(z-x) + yz\}$.

3. Solve $5\{x - 2[x - 3(x-1)]\} = 70$.

4. Divide $x^4 + x^3 - 24x^2 - 35x + 57$ by $x^2 + 2x - 3$.

5. Find the factors of

(i.) $(a+b)^2 - 121$; (ii.) $a^4 - b^4$; (iii.) $x^2 - 5x - 14$.

6. Find the H. C. F. of $a^3 - 2a^2 + 1$ and $2a^3 + a^2 + 4a - 7$.

7. A man being asked his age said: "Ten years ago I was five times as old as my son, but 20 years hence, I shall be only twice as old as he." How old was he?

8. Solve (i.) $6 - \frac{x-1}{2} - \frac{x-2}{3} = \frac{3-x}{4}$;

(ii.) $\frac{4x-9}{27} - \frac{x-3}{4} = \frac{5x-3}{6} - \frac{x+6}{2}$.

9. By how much does $y^3 - 3y^2 + 8y + 9$ exceed $y - 4y^2 + 6 - y^3$?

10. Show that

$$(ax+by)^2 + (ay-bx)^2 + c^2x^2 + c^2y^2 = (x^2+y^2)(a^2+b^2+c^2).$$

11. Solve (i.) $\frac{2x-1}{3} - \frac{3x-2}{4} = \frac{5x-4}{6} - \frac{7x+6}{12}$;

(ii.) $(3x-1)^2 + (4x-2)^2 = (5x-3)^2$.

12. If $x = 1$, $y = 2$, $z = 3$, find the value of

$$(x-y)[(x+z) + (y-z)] - x^2 + y(y+z) - zy.$$

13. Find the factors of (i.) $a^2 + 17ab + 60b^2$; (ii.) $10a^2 + 79a - 8$.

14. Simplify (i.) $\frac{ac}{a^2 - 4y^2} + \frac{bd}{ac + 2cy}$;

(ii.) $\frac{a}{ab+b^2} + \frac{b}{a^2-ab} + \frac{a}{a^2-b^2}$.

15. Find the L. C. M. of $x^3 + 6x^2 + 11x + 6$ and $x^3 - 7x + 6$.

16. The difference between the numerator and the denominator of a proper fraction is 8, and if each be increased by 17, the fraction becomes equal to $\frac{3}{4}$: find it.

17. Solve (i.) $20(7x+4) - 18(3x+4) - 5 = 25(x+5)$;

(ii.) $\frac{1}{3}(x+1) + \frac{1}{4}(x+3) = \frac{1}{5}(x+4) + 16$.

18. Divide (i.) $2x(x^2-1)(x+2)$ by $x^2 + x - 2$;

(ii.) $5x(x-11)(x^2 - x - 156)$ by $x^3 + x^2 - 132x$.

19. A boy is one-sixth the age of his father, and five years older than his sister; the united ages of all three being 51, how old is each?

20. Find the continued product of x^2+ax+a^2, x^2-ax+a^2, $x^4-a^2x^2+a^4$.

21. Show without actual division that $x-3$ is a factor of the expression x^3-2x^2-5x+6.

22. Simplify (i.) $\dfrac{x}{9}+\dfrac{2}{3}+\dfrac{4}{x-6}-\dfrac{2x}{3(x-6)}$;

(ii.) $\dfrac{2}{a-x}-\dfrac{1}{2a-x}+\dfrac{1}{x}$.

23. Find the H. C. F. of $2a^3+a^2-a-2$ and $a^5-a^3-2a^2+2a$ by the usual method. Is the work shortened by proceeding as in Art. 119? Show that the square of the H. C. F. is contained in the second expression.

24. Divide x^6+19x^3-216 by $(x^2-3x+9)(x-2)$.

25. Simplify (i.) $\dfrac{p+2}{2}-\dfrac{p}{p+2}-\dfrac{p^3-2p^2}{2p^2-8}$; (ii.) $\dfrac{3}{x^2-4}+\dfrac{1}{(x-2)^2}$.

26. Find the factors of (i.) $14a^2-11a-15$; (ii.) $a^4+5a^2b^2+9b^4$.

27. Of a party 5 more than one-third are Americans, 7 less than one-half are Englishmen, and the remainder, 8 in number, are Germans: find the number in the party.

28. Solve (i.) $2(5x-2)-3(5x-8)=5(x+1)-(2x-11)$;

(ii.) $\dfrac{2x-9}{27}+\dfrac{x}{18}-\dfrac{x-3}{4}=8\tfrac{1}{3}-x$.

29. Simplify (i.) $\dfrac{1}{a(x^2+a^2)}-\dfrac{1}{x(x+a)^2}$; (ii) $\dfrac{1+x+x^2}{1-x^3}+\dfrac{x-x^2}{(1-x)^3}$.

30. Find three numbers whose sum is 21, and of which the greatest exceeds the least by 4, and the middle one is half the sum of the other two.

31. Employ the Factor Theorem in finding the H. C. F. of

$$a^3-2a^2+1 \text{ and } 2a^3+a^2+4a-7.$$

32. Show that $\dfrac{3(x^2+x-2)}{x^2-x-2}-\dfrac{3(x^2-x-2)}{x^2+x-2}-\dfrac{8x}{x^2-4}=\dfrac{4x}{x^2-1}$.

33. Two trains go from P to Q by different routes, one of which is 15 miles longer than the other. A train on the shorter route takes 6 hours, and a train on the longer, travelling 10 miles less per hour, takes $8\tfrac{1}{2}$ hours. Find length of each route.

34. Find the factors of

(i.) $4x^2 - 4xy - 15y^2$; (ii.) $9x^4 - 82x^2y^2 + 9y^4$.

35. Solve $\dfrac{17-3x}{5} - \dfrac{2+4x}{3} = \dfrac{14+7x}{3} + 5 - 6x$.

36. The number of months in the age of a man on his birthday in 1875 was exactly half of the number denoting the year in which he was born. In what year was he born?

37. Simplify (i.) $\dfrac{2x-7}{(x-3)^2} - \dfrac{2(x+2)}{x^2-9}$; (ii.) $\dfrac{1}{2x^2-\frac{1}{2}} + \dfrac{1}{(2x+1)^2}$.

38. Divide $x^8+x^6y^2+x^4y^4+x^2y^6+y^8$ by $x^4-x^3y+x^2y^2-xy^3+y^4$ and find the value of the quotient when $x=0$ and $y=1$.

39. Simplify (i.) $\dfrac{1-x}{1-x+x^2} - \dfrac{\frac{1}{x}\left(\frac{1}{x}-2\right)}{\frac{1}{x^3}+1}$;

(ii.) $\dfrac{x}{1+\dfrac{x}{1-x+\dfrac{x}{1+x}}} \div \dfrac{1+x+x^2}{1+3x+3x^2+2x^3}$.

40. A regiment has sufficient food for m days; but if it were reinforced by p men, would have food enough for n days only. Find the number of men in the regiment.

41. Solve (i.) $\dfrac{x-\frac{1}{b}}{d} + \dfrac{x-\frac{1}{c}}{b} + \dfrac{x-\frac{1}{d}}{c} = 0$;

(ii.) $\dfrac{x-a}{b} + \dfrac{x+b}{a} = \dfrac{a+b}{a}$.

42. Simplify $(1+a)^2 \div \left\{1 + \dfrac{a}{1-a+\dfrac{a}{1+a+a^2}}\right\}$.

CHAPTER XVII.

SIMULTANEOUS EQUATIONS.

167. Consider the equation $2x + 5y = 23$, which contains *two* unknown quantities.

From this we get $$y = \frac{23 - 2x}{5} \quad \ldots \ldots \quad (1).$$

Now for every value we give to x there will be one corresponding value of y. Thus we shall be able to find as many pairs of values as we please which satisfy the given equation. Such an equation is called *indeterminate.*

For instance, if $x = 1$, then from (1) $y = \frac{21}{5}$.

Again, if $x = -2$, then $y = \frac{27}{5}$; and so on.

But if also we have a second equation of the same kind expressing a different relation between x and y, such as

$$3x + 4y = 24,$$

we have from this $$y = \frac{24 - 3x}{4} \quad \ldots \ldots \quad (2).$$

If now we seek values of x and y which satisfy *both* equations, the values of y in (1) and (2) must be identical.

Therefore $$\frac{23 - 2x}{5} = \frac{24 - 3x}{4}.$$

Multiplying across, $$92 - 8x = 120 - 15x;$$
$$7x = 28;$$
$$\therefore \ x = 4.$$

Substituting this value in equation (1), we have

$$y = \frac{23 - 2x}{5} = \frac{23 - 8}{5} = 3.$$

Thus, if both equations are to be satisfied by the *same* values of x and y, there is only one solution possible.

168. Definition. When two or more equations are satisfied by the same values of the unknown quantities, they are called **simultaneous equations.**

169. In the example already worked, we have used the method of solution which best illustrates the meaning of the term *simultaneous equations;* but in practice it will be found that this is rarely the readiest mode of solution. It must be borne in mind that since the two equations are simultaneously true, *any* equation formed by combining them will be satisfied by the values of x and y which satisfy the original equations. Our object will always be to obtain an equation which involves *one only* of the unknown quantities.

170. The process by which we cause either of the unknown quantities to disappear is called **elimination.** It may be effected in different ways, but three methods are in general use: (1) by **Addition** or **Subtraction**; (2) by **Substitution**; and (3) by **Comparison.**

ELIMINATION BY ADDITION OR SUBTRACTION.

171. Ex. 1. Solve $7x + 2y = 47$ . . . . (1),

$5x - 4y = 1$ . . . . (2).

Here it will be more convenient to eliminate y.

Multiplying (1) by 2, $14x + 4y = 94$,

and from (2) $5x - 4y = 1$;

adding, $19x = 95$;

$\therefore x = 5$.

To find y, substitute this value of x in *either* of the given equations.

Thus from (1) $35 + 2y = 47$;

$$\left.\begin{aligned} \therefore y &= 6, \\ x &= 5. \end{aligned}\right\}$$

and

In this solution we eliminated y by *addition*.

Ex. 2. Solve $3x + 7y = 27$ . . . . (1),

$5x + 2y = 16$ . . . . (2).

To eliminate x we multiply (1) by 5 and (2) by 3, so as to make the coefficients of x in both equations equal. This gives

$$15x + 35y = 135,$$

$$15x + 6y = 48;$$

subtracting, $$29y = 87;$$

$$\therefore\ y = 3.$$

To find x, substitute this value of y in *either* of the given equations.

Thus from (1) $$3x + 21 = 27;$$

$$\left.\begin{aligned}\therefore\ x &= 2,\\ \text{and}\quad y &= 3.\end{aligned}\right\}$$

In this solution we eliminated x by *subtraction*.

Rule. *Multiply, when necessary, in such a manner as to make the coefficients of the unknown quantity to be eliminated equal in both equations. Add the resulting equations if these coefficients are unlike in sign; subtract if like in sign.*

ELIMINATION BY SUBSTITUTION.

172. Ex. Solve $2x - 5y = 1$ . . . . (1),

$7x + 3y = 24$ . . . . (2).

Transposing $-5y$ in (1), and dividing by 2, we obtain

$$x = \frac{5y+1}{2}.$$

Substituting this value of x in (2) gives

$$7\left(\frac{5y+1}{2}\right) + 3y = 24.$$

Whence $$35y + 7 + 6y = 48,$$

and $$41y = 41;$$

$$\therefore\ y = 1.$$

This value substituted in *either* (1) or (2) gives

$$x = 3.$$

Rule. *From one of the equations, find the value of the unknown quantity to be eliminated in terms of the other and known quantities; then substitute this value for that quantity in the other equation, and reduce.*

ELIMINATION BY COMPARISON.

173. Ex. Solve $x + 15y = 53 \quad \ldots \quad (1),$

$y + 3x = 27 \quad \ldots \quad (2).$

From (1) $x = 53 - 15y,$

and from (2) $x = \frac{27 - y}{3}.$

Placing these values of x equal to each other, we have

$$53 - 15y = \frac{27 - y}{3}.$$

Whence $159 - 45y = 27 - y,$

and $44y = 132;$

$$\therefore\ y = 3.$$

Substituting this value in *either* (1) or (2) gives

$$x = 8.$$

Rule. *From each equation find the value of the unknown quantity to be eliminated in terms of the other and known quantities; then form an equation with these values, and reduce.*

EXAMPLES XVII. a.

Solve the equations:

1. $3x + 4y = 10,$ $4x + y = 9.$

2. $x + 2y = 13,$ $3x + y = 14.$

3. $4x + 7y = 29,$ $x + 3y = 11.$

4. $2x - y = 9,$ $3x - 7y = 19.$

5. $5x + 6y = 17,$ $6x + 5y = 16.$

6. $2x + y = 10,$ $7x + 8y = 53.$

7. $8x - y = 34,$ $x + 8y = 53.$

8. $15x + 7y = 29,$ $9x + 15y = 39.$

9. $14x - 3y = 39,$ $6x + 17y = 35.$

10. $28x - 23y = 33,$ $63x - 25y = 101.$

11. $35x + 17y = 86,$ $56x - 13y = 17.$

12. $15x + 77y = 92,$ $55x - 33y = 22.$

13. $5x - 7y = 0,$ $7x + 5y = 74.$

14. $21x - 50y = 60,$ $28x - 27y = 199.$

15. $39x - 8y = 99,$ $52x - 15y = 80.$

16. $5x = 7y - 21,$ $21x - 9y = 75.$

17. $6y - 5x = 18,$ $12x - 9y = 0.$

18. $8x = 5y,$ $13x = 8y + 1.$

19. $3x = 7y,$ $12y = 5x - 1.$

20. $19x + 17y = 0,$ $2x - y = 53.$

21. $93x + 15y = 123,$ $15x + 93y = 201.$

174. We add a few cases in which, before proceeding to solve, it will be necessary to simplify the equations.

Ex. 1. Solve $5(x+2y)-(3x+11y)=14 \quad \ldots\ldots (1),$

$$7x-9y-3(x-4y)=38 \quad \ldots\ldots (2).$$

From (1) $5x+10y-3x-11y=14;$

$$\therefore\ 2x-y=14 \quad \ldots\ldots (3).$$

From (2) $7x-9y-3x+12y=38;$

$$\therefore\ 4x+3y=38 \quad \ldots\ldots (4).$$

From (3) $6x-3y=42;$

and hence we may find $x=8$, and $y=2$.

Ex. 2. Solve $3x-\dfrac{y-5}{7}=\dfrac{4x-3}{2} \quad \ldots\ldots (1),$

$$\frac{3y+4}{5}-\tfrac{1}{3}(2x-5)=y \quad \ldots\ldots (2).$$

Clear of fractions. Thus

from (1) $42x-2y+10=28x-21;$

$$\therefore\ 14x-2y=-31 \quad \ldots\ldots (3).$$

From (2) $9y+12-10x+25=15y;$

$$\therefore\ 10x+6y=37 \quad \ldots\ldots (4).$$

Eliminating y from (3) and (4), we find that

$$x=-\tfrac{14}{13}.$$

Eliminating x from (3) and (4), we find that

$$y=\tfrac{207}{26}.$$

Note. Sometimes, as in the present instance, the value of the second unknown is more easily found by elimination than by substituting the value of the unknown already found.

EXAMPLES XVII. b.

1. $\dfrac{2x}{3}+y=16,$ $\quad x+\dfrac{y}{4}=14.$

2. $\dfrac{x}{5}+\dfrac{y}{2}=5,$ $\quad x-y=4.$

3. $\dfrac{5x}{6}-y=3,$ $\quad x-\dfrac{5y}{6}=8.$

4. $x-y=5,$ $\quad \dfrac{x}{4}-\dfrac{y}{5}=2.$

5. $\dfrac{x}{9}+\dfrac{y}{7}=10,$ $\quad \dfrac{x}{3}+y=50.$

6. $x=3y,$ $\quad \dfrac{x}{3}+y=34.$

7. $\frac{2}{5}x - \frac{1}{12}y = 3,$
$4x - y = 20.$

8. $\frac{1}{2}x - \frac{1}{5}y = 4,$
$\frac{1}{7}x + \frac{1}{15}y = 3.$

9. $2x + y = 0,$
$\frac{1}{2}y - 3x = 8.$

10. $\frac{x}{7} + \frac{y}{5} = 1\frac{3}{7},$
$x + \frac{y}{3} = 4\frac{2}{3}.$

11. $3x - 7y = 0,$
$\frac{2}{7}x + \frac{5}{3}y = 7.$

12. $\frac{x}{5} - \frac{y}{4} = 0,$
$3x + \frac{1}{2}y = 17.$

13. $\frac{3x-1}{2} - \frac{y}{4} = \frac{7}{2},$
$x + 3y = 9.$

14. $\frac{x}{3} + \frac{y}{4} = 3x - 7y - 37 = 0.$

15. $\frac{x+1}{10} = \frac{3y-5}{2} = \frac{x-y}{8}.$

SIMULTANEOUS EQUATIONS INVOLVING THREE UNKNOWN QUANTITIES.

175. In order to solve simultaneous equations which contain two unknown quantities we have seen that we must have two equations. Similarly, we find that in order to solve simultaneous equations which contain three unknown quantities we must have three equations.

Rule. *Eliminate one of the unknowns from any pair of the equations, and then eliminate the same unknown from another pair. Two equations involving two unknowns are thus obtained, which may be solved by the rules already given. The remaining unknown is then found by substituting in any one of the given equations.*

Ex. 1. Solve
$$6x + 2y - 5z = 13 \quad \ldots \ldots \quad (1),$$
$$3x + 3y - 2z = 13 \quad \ldots \ldots \quad (2),$$
$$7x + 5y - 3z = 26 \quad \ldots \ldots \quad (3).$$

Choose y as the unknown to be eliminated.

Multiply (1) by 3 and (2) by 2,
$$18x + 6y - 15z = 39,$$
$$6x + 6y - 4z = 26;$$
subtracting,
$$12x - 11z = 13 \quad \ldots \ldots \quad (4).$$

Again, multiply (1) by 5 and (3) by 2,
$$30x + 10y - 25z = 65,$$
$$14x + 10y - 6z = 52;$$
subtracting,
$$16x - 19z = 13 \quad \ldots \ldots \quad (5).$$

Multiply (4) by 4 and (5) by 3,

$$48x - 44z = 52,$$
$$48x - 57z = 39;$$

subtracting, $13z = 13;$

$$\left.\begin{aligned} \therefore\ z &= 1, \\ x &= 2, \\ y &= 3. \end{aligned}\right\}$$

and from (4) $x = 2$, from (1) $y = 3$.

NOTE. After a little practice the student will find that the solution may often be considerably shortened by a suitable combination of the proposed equations. Thus, in the present instance, by adding (1) and (2) and subtracting (3) we obtain $2x - 4z = 0$, or $x = 2z$. Substituting in (1) and (2), we have two easy equations in y and z.

Ex. 2. Solve $\dfrac{x}{2} - 1 = \dfrac{y}{6} + 1 = \dfrac{z}{7} + 2,$

$$\frac{y}{3} + \frac{z}{2} = 13.$$

From the equation $\dfrac{x}{2} - 1 = \dfrac{y}{6} + 1,$

we have $3x - y = 12 \quad \ldots\ldots\ldots \quad (1).$

Also, from the equation $\dfrac{x}{2} - 1 = \dfrac{z}{7} + 2,$

we have $7x - 2z = 42 \quad \ldots\ldots\ldots \quad (2).$

And, from the equation $\dfrac{y}{3} + \dfrac{z}{2} = 13,$

we have $2y + 3z = 78 \quad \ldots\ldots\ldots \quad (3).$

Eliminating z from (2) and (3), we have

$$21x + 4y = 282;$$

and from (1) $12x - 4y = 48;$

whence $x = 10$, $y = 18$. Also by substitution in (2) we obtain $z = 14$.

Ex. 3. Consider the equations

$$5x - 3y - z = 6 \quad \ldots\ldots\ldots \quad (1),$$
$$13x - 7y + 3z = 14 \quad \ldots\ldots\ldots \quad (2),$$
$$7x - 4y = 8 \quad \ldots\ldots\ldots \quad (3).$$

Multiplying (1) by 3 and adding to (2), we have

$$28x - 16y = 32,$$

or $7x - 4y = 8.$

Thus the combination of equations (1) and (2) leads us to an equation which is identical with (3), and so to find x and y we have but a single equation $7x - 4y = 8$, the solution of which is indeterminate. [Art. 167.]

In this and similar cases the anomaly arises from the fact that the equations are not *independent;* in other words, one equation is deducible from the others, and therefore contains no *relation* between the unknown quantities which is not already implied in the other equations.

EXAMPLES XVII. c.

1. $x + 2y + 2z = 11,\quad 2x + y + z = 7,\quad 3x + 4y + z = 14.$

2. $x + 3y + 4z = 14,\quad x + 2y + z = 7,\quad 2x + y + 2z = 2.$

3. $x + 4y + 3z = 17,\quad 3x + 3y + z = 16,\quad 2x + 2y + z = 11.$

4. $3x - 2y + z = 2,\quad 2x + 3y - z = 5,\quad x + y + z = 6.$

5. $2x + y + z = 16,\quad x + 2y + z = 9,\quad x + y + 2z = 3.$

6. $x - 2y + 3z = 2,\quad 2x - 3y + z = 1,\quad 3x - y + 2z = 9.$

7. $3x + 2y - z = 20,\quad 2x + 3y + 6z = 70,\quad x - y + 6z = 41.$

8. $2x + 3y + 4z = 20,\quad 3x + 4y + 5z = 26,\quad 3x + 5y + 6z = 31.$

9. $3x - 4y = 6z - 16,\quad 4x - y - z = 5,\quad x = 3y + 2(z - 1).$

10. $5x + 2y = 14,\quad y - 6z = -15,\quad x + 2y + z = 0.$

11. $x - \dfrac{y}{5} = 6,\quad y - \dfrac{z}{7} = 8,\quad z - \dfrac{x}{2} = 10.$

12. $\dfrac{y+z}{4} = \dfrac{z+x}{3} = \dfrac{x+y}{2},\quad x + y + z = 27.$

13. $\dfrac{y-z}{3} = \dfrac{y-x}{2} = 5z - 4x,\quad y + z = 2x + 1.$

14. $2x + 3y = 5,\quad 2z - y = 1,\quad 7x - 9z = 3.$

15. $\frac{1}{2}(x + z - 5) = y - z = 2x - 11 = 9 - (x + 2z).$

16. $x + 20 = \dfrac{3y}{2} + 10 = 2z + 5 = 110 - (y + z).$

176. Definition. If the product of two quantities be equal to unity, each is said to be the reciprocal of the other. Thus if $ab = 1$, a and b are **reciprocals**. They are so called because $a=\frac{1}{b}$, and $b=\frac{1}{a}$; and consequently a is related to b exactly as b is related to a.

The reciprocals of x and y are $\frac{1}{x}$ and $\frac{1}{y}$ respectively, and in solving the following equations we consider $\frac{1}{x}$ and $\frac{1}{y}$ as the unknown quantities.

Ex. 1. Solve
$$\frac{8}{x}-\frac{9}{y}=1 \quad \ldots\ldots \quad (1),$$
$$\frac{10}{x}+\frac{6}{y}=7 \quad \ldots\ldots \quad (2).$$

Multiply (1) by 2 and (2) by 3; thus
$$\frac{16}{x}-\frac{18}{y}=2,$$
$$\frac{30}{x}+\frac{18}{y}=21;$$
adding,
$$\frac{46}{x}=23;$$
multiplying across,
$$46=23x,$$
$$\therefore x=2;$$
and by substituting in (1), $y=3$.

Ex. 2. Solve
$$\frac{1}{2x}+\frac{1}{4y}-\frac{1}{3z}=\frac{1}{4} \quad \ldots\ldots \quad (1),$$
$$\frac{1}{x}=\frac{1}{3y} \quad \ldots\ldots \quad (2),$$
$$\frac{1}{x}-\frac{1}{5y}+\frac{4}{z}=2\tfrac{2}{15} \quad \ldots\ldots \quad (3);$$

clearing of fractional coefficients, we obtain

from (1)
$$\frac{6}{x}+\frac{3}{y}-\frac{4}{z}=3 \quad \ldots\ldots \quad (4),$$
from (2)
$$\frac{3}{x}-\frac{1}{y}=0 \quad \ldots\ldots \quad (5),$$
from (3)
$$\frac{15}{x}-\frac{3}{y}+\frac{60}{z}=32 \quad \ldots\ldots \quad (6).$$

Multiply (4) by 15 and add the result to (6); we have

$$\frac{105}{x}+\frac{42}{y}=77;$$

dividing by 7, $$\frac{15}{x}+\frac{6}{y}=11 \quad \ldots\ldots\ldots \quad (7);$$

from (5) $$\frac{18}{x}-\frac{6}{y}=0;$$

adding, $$\frac{33}{x}=11;$$

$$\therefore\ x=3,$$

from (5) $$y=1,$$

from (4) $$z=2.$$

EXAMPLES XVII. d.

1. $\dfrac{5}{x}+\dfrac{6}{y}=3,$
   $\dfrac{15}{x}+\dfrac{3}{y}=4.$

2. $\dfrac{6}{x}-\dfrac{7}{y}=2,$
   $\dfrac{2}{x}+\dfrac{14}{y}=3.$

3. $\dfrac{12}{x}-\dfrac{4}{y}=2.$
   $\dfrac{3}{x}-\dfrac{2}{y}=0.$

4. $\dfrac{5}{x}+\dfrac{16}{y}=79,$
   $\dfrac{16}{x}-\dfrac{1}{y}=44.$

5. $\dfrac{21}{x}+\dfrac{12}{y}=5,$
   $\dfrac{1}{y}-\dfrac{1}{x}=\dfrac{1}{42}.$

6. $\dfrac{5}{x}+\dfrac{3}{y}=30,$
   $\dfrac{9}{x}-\dfrac{5}{y}=2.$

7. $\dfrac{8}{x}-\dfrac{9}{y}=7,$
   $6\left(\dfrac{1}{x}+\dfrac{1}{y}\right)=1.$

8. $\dfrac{25}{x}+\dfrac{24}{y}=1,$
   $20\left(\dfrac{2}{x}+\dfrac{3}{y}\right)=7.$

9. $\dfrac{4}{x}+\dfrac{27}{y}=42,$
   $\dfrac{14}{x}-\dfrac{15}{y}=1.$

10. $\dfrac{3}{x}+\dfrac{5}{y}=\dfrac{8}{15},$
    $9y-22x=\dfrac{3xy}{25}.$

11. $\dfrac{1}{4x}+\dfrac{1}{3y}=2,$
    $\dfrac{1}{y}-\dfrac{1}{2x}=1.$

12. $2y-x=4xy,$
    $\dfrac{4}{y}-\dfrac{3}{x}=9.$

13. $\dfrac{1}{x}-\dfrac{2}{y}+4=0,$
    $\dfrac{1}{y}-\dfrac{1}{z}+1=0,$
    $\dfrac{2}{z}+\dfrac{3}{x}=14.$

14. $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}=36,$
    $\dfrac{1}{x}+\dfrac{3}{y}-\dfrac{1}{z}=28,$
    $\dfrac{1}{x}+\dfrac{1}{3y}+\dfrac{1}{2z}=20.$

LITERAL SIMULTANEOUS EQUATIONS.

177. Ex. 1. Solve $ax + by = c$ . . . . (1),

$a'x + b'y = c'$ . . . . (2).

The notation here first used is one that the student will frequently meet with in the course of his reading. In the first equation we choose certain letters as the coefficients of x and y, and we choose *corresponding letters with accents* to denote corresponding quantities in the second equation. There is no necessary connection between the values of a and a', read "a and a prime," and they are as different as a and b; but it is often convenient to use the same letter thus slightly varied to mark some common meaning of such letters, and thereby assist the memory. Thus a and a' have a common property as being coefficients of x; b, b' as being coefficients of y.

Sometimes instead of accents letters are used with a *suffix*, such as a_1, a_2, a_3; b_1, b_2, b_3, etc., read "a sub one, a sub two," etc.

To return to the equation $ax + by = c$ . . . . (1),

$a'x + b'y = c'$ . . . . (2).

Multiply (1) by b' and (2) by b. Thus

$$ab'x + bb'y = b'c,$$

$$a'bx + bb'y = bc';$$

by subtraction, $$(ab' - a'b)x = b'c - bc';$$

$$\therefore\ x = \frac{b'c - bc'}{ab' - a'b} \quad \ldots\ldots\ (3).$$

As previously explained in Art. 171, we might obtain y by substituting this value of x in *either* of the equations (1) or (2); but y is more conveniently found by eliminating x, as follows:

Multiplying (1) by a' and (2) by a, we have

$$aa'x + a'by = a'c,$$

$$aa'x + ab'y = ac';$$

by subtraction, $$(a'b - ab')y = a'c - ac';$$

$$\therefore\ y = \frac{a'c - ac'}{a'b - ab'}.$$

or, changing signs in the terms of the numerator and denominator so as to have the same denominator as in (3),

$$y=\frac{ac'-a'c}{ab'-a'b}, \text{ and } x=\frac{b'c-bc'}{ab'-a'b}.$$

Ex. 2. Solve
$$\frac{x-a}{c-a}+\frac{y-b}{c-b}=1 \quad \ldots\ldots\ldots \quad (1),$$
$$\frac{x+a}{c}+\frac{y-a}{a-b}=\frac{a}{c} \quad \ldots\ldots\ldots \quad (2).$$

From (1) by clearing of fractions, we have

$$x(c-b)-a(c-b)+y(c-a)-b(c-a)=(c-a)(c-b),$$
$$x(c-b)+y(c-a)=ac-ab+bc-ab+c^2-ac-bc+ab,$$
$$x(c-b)+y(c-a)=c^2-ab \quad \ldots\ldots \quad (3).$$

Again, from (2), we have

$$x(a-b)+a(a-b)+cy-ca=a(a-b),$$
$$x(a-b)+cy=ac \quad \ldots\ldots\ldots \quad (4).$$

Multiply (3) by c and (4) by $c-a$ and subtract,

$$x\{c(c-b)-(c-a)(a-b)\}=c^3-abc-ac(c-a),$$
$$x(c^2-ac+a^2-ab)=c(c^2-ab-ac+a^2);$$
$$\therefore x=c;$$

and therefore from (4) $\qquad y=b.$

EXAMPLES XVII. e.

1. $ax+by=l,$
   $bx+ay=m.$

2. $lx+my=n,$
   $px+qy=r.$

3. $ax=by,$
   $bx+ay=c.$

4. $ax+by=a^2,$
   $bx+ay=b^2.$

5. $x+ay=a',$
   $ax+a'y=1.$

6. $px-qy=r,$
   $rx-py=q.$

7. $\frac{x}{a}+\frac{y}{b}=\frac{1}{ab},$
   $\frac{x}{a'}-\frac{y}{b'}=\frac{1}{a'b'}.$

8. $\frac{x}{a}-\frac{y}{b}=0,$
   $bx+ay=4ab.$

9. $\frac{3x}{a}+\frac{2y}{b}=3,$
   $\frac{9x}{a}-\frac{6y}{b}=3.$

10. $qx-rb=p(a-y),$
    $\frac{qx}{a}+r=p\left(1+\frac{y}{b}\right).$

11. $\frac{x}{m}+\frac{y}{m'}=1,$
    $\frac{x}{m'}-\frac{y}{m}=1.$

12. $px+qy=0,$
    $lx+my=n.$

13. $(a-b)x=(a+b)y,$
    $x+y=c.$

14. $(a-b)x+(a+b)y=2a^2-2b^2,$
$(a+b)x-(a-b)y=4ab.$

15. $\frac{x}{a}+\frac{y}{b}=1,\ \frac{x}{3a}+\frac{y}{6b}=\frac{2}{3}.$

16. $\frac{x}{a}+\frac{y}{b}=2,\ \frac{x}{a'}=\frac{y}{b'}.$

17. $\frac{x}{a}-\frac{y}{b}=1,\ \frac{x}{b}+\frac{y}{a}=\frac{a}{b}.$

18. $\frac{x}{c+d}+\frac{y}{c-d}=2,$
$cx-dy=c^2+d^2.$

19. $\frac{a}{bx}+\frac{b}{ay}=a+b,$
$\frac{b}{x}+\frac{a}{y}=a^2+b^2.$

20. $ay+bx=2xy,$
$cy+dx=3xy.$

21. $\frac{x+y}{x-y}=\frac{l}{m-n},$
$\frac{x+m}{y+m}=\frac{l+m}{l+n}.$

22. $\frac{m}{l}x+\frac{l}{m}y=\left(\frac{1}{l}+\frac{1}{m}\right)(m^2+l^2),$
$(x+y)(m^2+l^2)=2(m^3+l^3)+ml(x+y).$

23. $bx+cy=a+b,\ ax\left(\frac{1}{a-b}-\frac{1}{a+b}\right)+cy\left(\frac{1}{b-a}-\frac{1}{b+a}\right)=\frac{2a}{a+b}.$

24. $(a-b)x+(a+b)y=2(a^2-b^2),\ ax-by=a^2+b^2.$

CHAPTER XVIII.

Problems leading to Simultaneous Equations.

178. In the Examples discussed in the last chapter we have seen that it is essential to have as many equations as there are unknown quantities to determine. Consequently in the solution of problems which give rise to simultaneous equations, it will always be necessary that the statement of the question should contain as many *independent conditions* as there are quantities to be determined.

Ex. **1.** Find two numbers whose difference is 11, and one-fifth of whose sum is 9.

Let x represent the greater number, y the less.

Then $$x - y = 11 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Also, $$\frac{x+y}{5} = 9,$$

or $$x + y = 45 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

By addition, $2x = 56$; and by subtraction, $2y = 34$.

The numbers are therefore 28 and 17.

Ex. **2.** If 15 lbs. of tea and 10 lbs. of coffee together cost \$15.50, and 25 lbs. of tea and 13 lbs. of coffee together cost \$24.55, find the price of each per pound.

Suppose a pound of tea to cost x cents and a pound of coffee to cost y cents.

Then from the question, we have

$$15x + 10y = 1550 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$25x + 13y = 2455 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Multiplying (1) by 5 and (2) by 3, we have

$$75x + 50y = 7750,$$
$$75x + 39y = 7365.$$

Subtracting, $$11y = 385,$$
$$y = 35.$$

And from (1) $15x + 350 = 1550.$

Whence $15x = 1200;$

$\therefore x = 80.$

Therefore the cost of a pound of tea is 80 cents, and the cost of a pound of coffee is 35 cents.

Ex. **3.** A person spent $6.80 in buying oranges at the rate of 3 for 10 cents, and apples at 15 cents a dozen; if he had bought five times as many oranges and a quarter of the number of apples, he would have spent $25.45. How many of each did he buy?

Let x represent the number of oranges and y the number of apples.

$$x \text{ oranges cost } \frac{10x}{3} \text{ cents,}$$

$$y \text{ apples cost } \frac{15y}{12} \text{ cents;}$$

$$\therefore \frac{10x}{3} + \frac{15y}{12} = 680 \quad \ldots \ldots \quad (1).$$

Again, $5x$ oranges cost $5x \times \frac{10}{3}$, or $\frac{50x}{3}$ cents, and $\frac{y}{4}$ apples cost $\frac{y}{4} \times \frac{15}{12}$, or $\frac{15y}{48}$ cents;

$$\therefore \frac{50x}{3} + \frac{15y}{48} = 2545 \quad \ldots \ldots \quad (2).$$

Multiply (1) by 5 and subtract (2) from the result;

then $\left(\frac{75}{12} - \frac{15}{48}\right)y = 855;$

or $\frac{285y}{48} = 855;$

$\therefore y = 144;$

and from (1) $x = 150.$

Thus there were 150 oranges and 144 apples.

Ex. **4.** If the numerator of a fraction is increased by 2 and the denominator by 1, it equals $\frac{5}{8}$; and if the numerator and denominator are each diminished by 1, it equals $\frac{1}{2}$: find the fraction.

Let x represent the numerator of the fraction, y the denominator: then the fraction is $\frac{x}{y}$.

From the first supposition, $\frac{x+2}{y+1} = \frac{5}{8} \quad \ldots \ldots \quad (1),$

from the second, $\frac{x-1}{y-1} = \frac{1}{2} \quad \ldots \ldots \quad (2).$

These equations give $x = 8$, $y = 15$.

Thus the fraction is $\frac{8}{15}$.

Ex. 5. The middle digit of a number between 100 and 1000 is zero, and the sum of the other digits is 11. If the digits be reversed, the number so formed exceeds the original number by 495. Find it.

Let x represent the digit in the units' place;

y represent the digit in the hundreds' place;

then, since the digit in the tens' place is 0, the number will be represented by $100y + x$. [Art. 84, Ex. 4.]

And if the digits are reversed, the number so formed will be represented by $100x + y$.

$$\therefore\ 100x + y - (100y + x) = 495,$$

or

$$100x + y - 100y - x = 495;$$

$$\therefore\ 99x - 99y = 495,$$

that is,

$$x - y = 5 \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Again, since the sum of the digits is 11, and the middle one is 0, we have

$$x + y = 11 \quad . \quad . \quad . \quad . \quad . \quad (2).$$

From (1) and (2) we find $x = 8$, $y = 3$.

Hence the number is 308.

EXAMPLES XVIII.

1. Find two numbers whose sum is 34, and whose difference is 10.

2. The sum of two numbers is 73, and their difference is 37: find the numbers.

3. One-third of the sum of two numbers is 14, and one-half of their difference is 4: find the numbers.

4. One-nineteenth of the sum of two numbers is 4, and their difference is 30: find the numbers.

5. Half the sum of two numbers is 20, and three times their difference is 18: find the numbers.

6. Six pounds of tea and eleven pounds of sugar cost \$5.65, and eleven pounds of tea and six pounds of sugar cost \$9.65. Find the cost of tea and sugar per pound.

7. Six horses and seven cows can be bought for \$250, and thirteen cows and eleven horses can be bought for \$461. What is the value of each animal?

8. A, B, C, D have \$290 between them; A has twice as much as C, and B has three times as much as D; also C and D together have \$50 less than A. Find how much each has.

9. A, B, C, D have \$270 between them; A has three times as much as C, and B five times as much as D; also A and B together have \$50 less than eight times what C has. Find how much each has.

10. Four times B's age exceeds A's age by twenty years, and one-third of A's age is less than B's age by two years: find their ages.

11. One-eleventh of A's age is greater by two years than one-seventh of B's, and twice B's age is equal to what A's age was thirteen years ago: find their ages.

12. In eight hours A walks twelve miles more than B does in seven hours; and in thirteen hours B walks seven miles more than A does in nine hours. How many miles does each walk per hour?

13. In eleven hours C walks $12\frac{1}{2}$ miles less than D does in twelve hours; and in five hours D walks $3\frac{1}{4}$ miles less than C does in seven hours. How many miles does each walk per hour?

14. Find a fraction such that if 1 be added to its denominator it reduces to $\frac{1}{2}$, and reduces to $\frac{3}{5}$ on adding 2 to its numerator.

15. Find a fraction which becomes $\frac{1}{2}$ on subtracting 1 from the numerator and adding 2 to the denominator, and reduces to $\frac{1}{3}$ on subtracting 7 from the numerator and 2 from the denominator.

16. If 1 be added to the numerator of a fraction it reduces to $\frac{1}{5}$; if 1 be taken from the denominator it reduces to $\frac{1}{7}$. Required the fraction.

17. If $\frac{2}{3}$ be added to the numerator of a certain fraction the fraction will be increased by $\frac{1}{21}$, and if $\frac{1}{2}$ be taken from its denominator the fraction becomes $\frac{2}{9}$: find it.

18. The sum of a number of two digits and of the number formed by reversing the digits is 110, and the difference of the digits is 6: find the numbers.

19. The sum of the digits of a number is 13, and the difference between the number and that formed by reversing the digits is 27: find the numbers.

20. A certain number of two digits is three times the sum of its digits, and if 45 be added to it the digits will be reversed: find the number.

21. A certain number between 10 and 100 is eight times the sum of its digits, and if 45 be subtracted from it the digits will be reversed: find the number.

22. A man has a number of silver dollars and dimes, and he observes that if the dollars were turned into dimes and the dimes into dollars he would gain \$2.70; but if the dollars were turned into half-dollars and the dimes into quarters he would lose \$1.30. How many of each had he?

23. In a bag containing black and white balls, half the number of white is equal to a third of the number of black; and twice the whole number of balls exceeds three times the number of black balls by four. How many balls did the bag contain?

24. A number consists of three digits, the right hand one being zero. If the left hand and middle digits be interchanged, the number is diminished by 180; if the left hand digit be halved and the middle and right hand digits be interchanged, the number is diminished by 454: find the number.

25. The wages of 10 men and 8 boys amount to $22.30; if 4 men together receive $3.40 more than 6 boys, what are the wages of each man and boy?

26. A grocer wishes to mix sugar at 8 cents a pound with another sort at 5 cents a pound to make 60 pounds to be sold at 6 cents a pound. What quantity of each must he take?

27. A traveller walks a certain distance; had he gone half a mile an hour faster, he would have walked it in four-fifths of the time; had he gone half a mile an hour slower, he would have been $2\frac{1}{2}$ hours longer on the road: find the distance.

28. A man walks 35 miles partly at the rate of 4 miles an hour, and partly at 5; if he had walked at 5 miles an hour when he walked at 4, and *vice versâ*, he would have covered 2 miles more in the same time: find the time he was walking.

29. Two persons, 27 miles apart, setting out at the same time are together in 9 hours if they walk in the same direction, but in 3 hours if they walk in opposite directions: find their rates of walking.

30. When a certain number of two digits is doubled, and increased by 10, the result is the same as if the number had been reversed, and doubled, and then diminished by 8; also the number itself exceeds 3 times the sum of its digits by 18: find the number.

31. If I lend a sum of money at 6 per cent, the interest for a certain time exceeds the loan by $100; but if I lend it at 3 per cent, for a fourth of the time, the loan exceeds its interest by $425. How much do I lend?

32. A takes 3 hours longer than B to walk 30 miles; but if he doubles his pace he takes 2 hours less time than B: find their rates of walking.

CHAPTER XIX.

Indeterminate and Impossible Problems. Negative Results. Meaning of $\frac{a}{0}, \frac{a}{\infty}, \frac{0}{0}, \frac{\infty}{\infty}$.

INDETERMINATE AND IMPOSSIBLE PROBLEMS.

179. By reference to Art. 167, it will be seen that a single equation involving two unknown quantities is satisfied by an indefinitely great number of sets of values of the unknowns involved, and that it is essential to have as many equations expressing different, or independent conditions, as there are unknown quantities to be determined. If the conditions of a problem furnish a *less* number of independent equations than quantities to be determined, the problem is said to be **indeterminate.** If, however, the conditions give us a *greater* number of independent equations than there are unknown quantities involved, the problem is **impossible.**

Suppose the problem furnishes

$$3x + y = 10,$$
$$2x + y = 5,$$
$$x + y = 3.$$

From (1) and (2) we obtain $x = 5$ and $y = -5$. From (2) and (3) we obtain $x = 2$ and $y = 1$. These values cannot all be true at the same time, hence the problem is *impossible.*

NEGATIVE RESULTS.

180. A *is* 40 years old, and B's age *is* three-fifths of A's. When *will* A be five times as old as B?

Let x represent the number of years that *will* elapse.

Then $$40 + x = 5(24 + x);$$

$$\therefore\ 40 + x = 120 + 5x,$$

or $$x = -20.$$

According to this analysis, A will be five times as old as B in -20 years. The meaning of this result ought to be at once evident to a thoughtful student. Were the result any positive number of years, we would simply count that number forward from the present time (represented by the word "is" in the problem); manifestly then the -20 years refers to past time. Hence the problem should read, "A is 40 years old, and B's age is three-fifths of A's. When *was* A five times as old as B?"

Suppose the problem read

A is 40 years old, and B's age is three-fifths of A's: find the time at which A's age is five times that of B.

Let us assume that x years *will* elapse.

Then $$40 + x = 5(24 + x);$$

$$\therefore\ x = -20.$$

Interpreting this result, we see that we should have assumed that x years *had* elapsed.

The student will notice that the word "will" in the first statement *suggested* that we should assume x as the number of years that *would* elapse, and that the negative result showed a fault in the enunciation of the problem; but that the problem, as given in the next discussion, *permitted* us to make one of two possible suppositions as to the nature of the unknown quantity, so that the negative result indicates simply a wrong choice.

Hence in the solution of problems involving equations of the first degree, *negative results* indicate

(1) *A fault in the enunciation of the problem*, or

(2) *A wrong choice between two possible suppositions, as to the nature of the unknown quantity, allowed by the problem.*

Generally it will be easy for the student to make such changes as will give an analogous possible problem.

EXAMPLES XIX.

Make such necessary changes in the statements of the following problems as will render them possible arithmetically.

1. A is 27 years old and B 15; in how many years will A be twice as old as B?

2. What are the two numbers whose difference is 50, and sum 40?

3. If to the sum of twice a certain number and $\frac{1}{5}$ of the same number 10 be added, the result is equal to twice the number.

4. A man loses \$400, and then finds that 6 times what he had at first is equal to 5 times what he has left.

5. What fraction is that which becomes $\frac{4}{7}$ when 1 is subtracted from its numerator, and $\frac{1}{2}$ when 1 is subtracted from its denominator?

6. A is to-day 25 years old, and B's age is $\frac{4}{5}$ of A's: find the date when A's age is twice that of B.

MEANING OF $\frac{a}{0}$, $\frac{a}{\infty}$, $\frac{0}{0}$, $\frac{\infty}{\infty}$.

181. Meaning of $\frac{a}{0}$. Consider the fraction $\frac{a}{x}$ in which the numerator a has a *certain fixed value*, and the denominator x is a *quantity subject to change;* then it is clear that the smaller x becomes, the larger does the value of the fraction $\frac{a}{x}$ become. For instance,

$$\frac{a}{\frac{1}{10}} = 10\,a, \quad \frac{a}{\frac{1}{1000}} = 1000\,a, \quad \frac{a}{\frac{1}{100000}} = 100000\,a.$$

By making the denominator x sufficiently small, the value of the fraction $\frac{a}{x}$ can be made as large as we please; that is, as the denominator x approaches to the value 0, the fraction becomes infinitely great. The symbol ∞ is used to express a quantity infinitely great, or more shortly *infinity.* The full verbal statement, given above, is sometimes written

$$\frac{a}{0} = \infty.$$

182. Meaning of $\frac{a}{\infty}$. If, in the fraction $\frac{a}{x}$, the denominator x gradually increases and finally becomes infinitely large, the fraction $\frac{a}{x}$ becomes infinitely small; that is, as the denominator of a fraction approaches to the value *infinity*, the fraction itself approaches to the value 0. This full verbal statement is sometimes written

$$\frac{a}{\infty}=0.$$

183. Meaning of $\frac{0}{0}$. The symbol $\frac{0}{0}$ may be indeterminate in *form* or in *fact*. Thus the value of $\frac{x^2-4}{x-2}$ when $x=2$ is $\frac{0}{0}$, but by putting the fraction in the form $\frac{(x+2)(x-2)}{x-2}$ we see that the expression is equivalent to $x+2$, which becomes 4 when $x=2$. Again, $\frac{x^3-a^3}{x-a}=\frac{0}{0}$ when $x=a$, but by putting the fraction in the form $\frac{(x-a)(x^2+xa+a^2)}{x-a}$ we see that the expression is equivalent to x^2+xa+a^2, or $3a^2$, when $x=a$. These fractions assumed the form $\frac{0}{0}$ under particular conditions, but it is evident that they do not necessarily have the same value.

On the other hand, the symbol $\frac{0}{0}$ may show that a value is really indeterminate. Thus, solving in the regular way the equations

$$x+y+2=0,$$
$$2x+2y+4=0,$$

we get $x=\frac{4-4}{2-2}=\frac{0}{0}$, and we can easily see that x can have any value whatever if we give y a value to suit, so that the value of x is indeterminate.

184. Meaning of $\frac{\infty}{\infty}$. Inasmuch as $\frac{1}{\infty}=0$, what is true of $\frac{0}{0}$ is equally so of $\frac{\infty}{\infty}$.

CHAPTER XX.

INVOLUTION.

185. DEFINITION. **Involution** is the general name for repeating an expression as a factor, so as to find its second, third, fourth, or any other power.

Involution may always be effected by actual multiplication. Here, however, we shall give some rules for writing at once

(1) any power of a monomial;

(2) the square and cube of any binomial;

(3) the square and cube of any multinomial;

(4) any power of a binomial expressed by a positive integer.

186. It is evident from the **Rule of Signs** that

(1) no **even power** of *any* quantity can be *negative;*

(2) any **odd power** of a quantity will have *the same sign* as the quantity itself.

NOTE. It is especially worthy of notice that the *square* of every expression, whether positive or negative, is *positive.*

INVOLUTION OF MONOMIALS.

187. From definition we have, by the rules of multiplication,

$$(a^2)^3 = a^2 \cdot a^2 \cdot a^2 = a^{2+2+2} = a^6.$$

$$(-x^3)^2 = (-x^3)(-x^3) = x^{3+3} = x^6.$$

$$(-a^5)^3 = (-a^5)(-a^5)(-a^5) = -a^{5+5+5} = -a^{15}.$$

$$(-3a^3)^4 = (-3)^4(a^3)^4 = 81a^{12}.$$

Hence we obtain the following rule for raising a simple expression to any proposed power:

Rule. (1) *Raise the coefficient to the required power by Arithmetic, and prefix the proper sign found by Art. 42.*

(2) *Multiply the index of every factor of the expression by the exponent of the power required.*

EXAMPLES. (1) $(-2x^2)^5 = -32x^{10}$.

(2) $(-3ab^3)^6 = 729a^6b^{18}$.

(3) $\left(\frac{2ab^3}{3x^2y}\right)^4 = \frac{16a^4b^{12}}{81x^8y^4}$.

It will be seen that in the last case the numerator and the denominator are operated upon separately.

EXAMPLES XX. a.

Write the square of each of the following expressions:

1. $3ab^3$. **2.** $5x^2y^5$. **3.** $-2abc^2$. **4.** $11b^2c^3$. **5.** $4xyz^3$. **6.** $-\frac{2}{3}a^2b^3$. **7.** $\frac{2x^2}{3y^3}$. **8.** $-\frac{4}{3x^2y}$. **9.** $-\frac{7ab}{3}$. **10.** $\frac{3a^2b^3}{4c^5x^4}$. **11.** $-2xy^2$. **12.** $-\frac{3a^5}{5x^3}$.

Write the cube of each of the following expressions:

13. $2ab^2$. **14.** $3x^3$. **15.** $-2a^7c^2$. **16.** $-3a^3b$. **17.** $\frac{1}{3y^2}$. **18.** $-\frac{3x^5}{5a^3}$. **19.** $7x^3y^4$. **20.** $-\frac{2}{3}a^5$.

Write the value of each of the following expressions:

21. $(3a^2b^3)^4$. **22.** $(-a^2x)^6$. **23.** $(-2x^3y)^5$. **24.** $\left(\frac{1}{2a^2}\right)^7$. **25.** $\left(\frac{3x^4}{2y^3}\right)^5$. **26.** $\left(\frac{2x^3}{3y}\right)^8$. **27.** $\left(-\frac{x^3}{3}\right)^7$. **28.** $\left(-\frac{2x^5}{3a^4}\right)^6$. **29.** $\left(-\frac{2a^2x^3}{5bc^2}\right)^4$.

TO SQUARE A BINOMIAL.

188. By multiplication we have

$$(a+b)^2 = (a+b)(a+b) = a^2 + 2ab + b^2 \quad . \; . \quad (1),$$
$$(a-b)^2 = (a-b)(a-b) = a^2 - 2ab + b^2 \quad . \; . \quad (2).$$

Rule I. *The square of the sum of two quantities is equal to the sum of their squares increased by twice their product.*

Rule II. *The square of the difference of two quantities is equal to the sum of their squares diminished by twice their product.*

Ex. 1.
$$\begin{aligned}(x + 2y)^2 &= x^2 + 2 \cdot x \cdot 2y + (2y)^2 \\ &= x^2 + 4xy + 4y^2.\end{aligned}$$

Ex. 2.
$$\begin{aligned}(2a^3 - 3b^2)^2 &= (2a^3)^2 - 2 \cdot 2a^3 \cdot 3b^2 + (3b^2)^2 \\ &= 4a^6 - 12a^3b^2 + 9b^4.\end{aligned}$$

189. These rules may sometimes be conveniently applied to find the squares of numerical quantities.

Ex. 1. The square of 1012
$$\begin{aligned}&= (1000 + 12)^2 \\ &= (1000)^2 + 2 \cdot 1000 \cdot 12 + (12)^2 \\ &= 1000000 + 24000 + 144 \\ &= 1024144.\end{aligned}$$

Ex. 2. The square of 98
$$\begin{aligned}&= (100 - 2)^2 \\ &= (100)^2 - 2 \cdot 100 \cdot 2 + (2)^2 \\ &= 10000 - 400 + 4 \\ &= 9604.\end{aligned}$$

TO SQUARE A MULTINOMIAL.

190. We may now extend the rules of Art. 188 thus:

$$\begin{aligned}(a + b + c)^2 &= \{(a + b) + c\}^2 \\ &= (a + b)^2 + 2(a + b)c + c^2 \quad \text{[Art. 188, Rule 1.]} \\ &= a^2 + b^2 + c^2 + 2ab + 2ac + 2bc.\end{aligned}$$

In the same way we may prove

$$(a - b + c)^2 = a^2 + b^2 + c^2 - 2ab + 2ac - 2bc$$

$$\begin{aligned}(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac \\ + 2ad + 2bc + 2bd + 2cd.\end{aligned}$$

In each instance we observe that the square consists of

(1) the sum of the squares of the several terms of the given expression;

(2) twice the sum of the products two and two of the several terms, taken with their proper signs; that is, in each

product the sign is + or − according as the quantities composing it have like or unlike signs.

NOTE. The *square terms* are always positive.

The same laws hold whatever be the number of terms in the expression to be squared.

Rule. *To find the square of any multinomial: to the sum of the squares of the several terms add twice the product (with the proper sign) of each term into each of the terms that follow it.*

Ex. 1. $$(x-2y-3z)^2 = x^2+4y^2+9z^2-2\cdot x\cdot 2y-2\cdot x\cdot 3z+2\cdot 2y\cdot 3z$$
$$= x^2+4y^2+9z^2-4xy-6xz+12yz.$$

Ex. 2. $$(1+2x-3x^2)^2 = 1+4x^2+9x^4+2\cdot 1\cdot 2x-2\cdot 1\cdot 3x^2-2\cdot 2x\cdot 3x^2$$
$$= 1+4x^2+9x^4+4x-6x^2-12x^3$$
$$= 1+4x-2x^2-12x^3+9x^4.$$

EXAMPLES XX. b.

Write the square of each of the following expressions:

1. $a+3b$.
2. $a-3b$.
3. $x-5y$.
4. $2x+3y$.
5. $3x-y$.
6. $3x+5y$.
7. $9x-2y$.
8. $5ab-c$.
9. $a-b-c$.
10. $a+b-c$.
11. $a+2b+c$.
12. $2a-3b+4c$.
13. $x^2-y^2-z^2$.
14. $xy+yz+zx$.
15. $x-y+a-b$.
16. $2x+3y+a-2b$.
17. $m-n-p-q$.
18. $\frac{a}{2}-2b+\frac{c}{4}$.
19. $\frac{a}{3}-3b-\frac{3}{2}$.
20. $\frac{2}{3}x^2-x+\frac{3}{2}$.

TO CUBE A BINOMIAL.

191. By actual multiplication, we have

$$(a+b)^3 = (a+b)(a+b)(a+b)$$
$$= a^3+3a^2b+3ab^2+b^3 \quad . \quad . \quad . \quad (1),$$
$$(a-b)^3 = a^3-3a^2b+3ab^2-b^3 \quad . \quad . \quad . \quad (2).$$

From these results we obtain the following rule: —

Rule. *To find the cube of any binomial: take the cube of the first term, three times the square of the first by the second, three times the first by the square of the second, and the cube of the last.*

If the binomial be the sum of the quantities, all signs will be $+$; if the difference of two quantities, the signs will be alternately $+$ and $-$, commencing with the first.

EXAMPLES XX. c.

Write the cube of each of the following expressions:

1. $x + a$.
2. $x - a$.
3. $x - 2y$.
4. $2ab - 3c$.
5. $x^2 + 4y^2$.
6. $4x^2 - 5y^2$.
7. $2a^3 - 3b^2$.
8. $5x^5 - 4y^4$.
9. $a - \frac{2b}{3}$.
10. $\frac{a}{3} + 2$.
11. $\frac{x^2}{3} - 3x$.
12. $\frac{a}{6} + 2x$.

TO CUBE ANY MULTINOMIAL.

192. Consider a trinomial:

$$\begin{aligned}(a+b+c)^3 &= [a+(b+c)]^3 \\ &= a^3 + 3a^2(b+c) + 3a(b+c)^2 + (b+c)^3 \\ &= a^3 + b^3 + c^3 + 3a^2(b+c) + 3b^2(a+c) \\ &\quad + 3c^2(a+b) + 6abc.\end{aligned}$$

Rule. *To cube a multinomial: take the cube of each term, three times the square of each term by every other term, and six times the product of every three different terms. The signs are determined by the law of signs for multiplication.*

EXAMPLES XX. d.

Write the cube of each of the following expressions:

1. $1 + x + x^2$.
2. $1 + x - x^2$.
3. $1 - 2x + 3x^2$.
4. $a + bx + x^2$.
5. $2a + bx - cx^2$.
6. $3x + 2x^2 - x^3$.
7. $\frac{x}{2} + \frac{x^2}{3} - x^3$.
8. $1 + x + x^2 + x^3$.
9. $2 - 3x + x^2 + 2x^3$.

TO RAISE A BINOMIAL TO ANY POWER EXPRESSED BY A POSITIVE INTEGER [BINOMIAL THEOREM].

193. By actual multiplication, we obtain the following identities:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3;$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

In these results, spoken of as expansions, we notice that:

(1) *The number of terms equals the index of the binomial plus one.*

(2) *The exponent of a in the first term is the same as the index of the binomial, and decreases by one in each succeeding term.*

(3) *The quantity b appears for the first time in the second term of the expansion with an exponent* 1, *and its exponent increases by one in each succeeding term.*

(4) *The coefficient of the first term is* 1.

(5) *The coefficient of the second term is the same as the index of the binomial.*

(6) *The coefficient of any term may be found by multiplying the coefficient of the preceding term by the exponent of a in that term, and dividing the result by the exponent of b plus* 1.

Ex. 1. Expand $(a+b)^6$.

$$\begin{array}{lllllll}
(a+b)^6 = a^6 & + a^5 & + a^4 & + a^3 & + a^2 & + a & \\
 & + b & + b^2 & + b^3 & + b^4 & + b^5 & + b^6 \\
\text{Coefficients,}\quad 1 & + 6 & + 15 & + 20 & + 15 & + 6 & + 1 \\
\hline
\text{Multiplying,}\quad x^6 & + 6a^5b & + 15a^4b^2 & + 20a^3b^3 & + 15a^2b^4 & + 6ab^5 & + b^6
\end{array}$$

Ex. 2. Expand $(a-2b^2)^4$.

$$\begin{array}{lllll}
(a-2b^2)^4 = [a+(-2b^2)]^4 & & & & \\
= a^4 & + a^3 & + a^2 & + a & \\
 & + (-2b^2) & + (-2b^2)^2 & + (-2b^2)^3 & + (-2b^2)^4 \\
\text{Coefficients,}\quad 1 & + 4 & + 6 & + 4 & + 1 \\
\hline
\text{Multiplying,}\quad a^4 & - 8a^3b & + 24a^2b^4 & - 32ab^6 & + 16a^8
\end{array}$$

Note. The student will observe that in the line of coefficients, terms at equal distances from the beginning and the end are equal.

194. The same method may be used in expanding any multinomial.

Ex. Expand $(a+2b-c)^3$.

$$\begin{array}{llll}
(a+2b-c)^3 = [(a+2b)+(-c)]^3 & & & \\
= (a+2b)^3 & + (a+2b)^2 & + (a+2b) & \\
 & + (-c) & + (-c)^2 & + (-c)^3 \\
\text{Coefficients,}\quad 1 & + 3 & + 3 & + 1 \\
\hline
\text{Multiplying,}\quad (a+2b)^3 & - 3(a+2b)^2c & + 3(a+2b)c^2 & - c^3
\end{array}$$

Performing the operations indicated, we have

$$a^3 + 6a^2b - 3a^2c + 12ab^2 - 12abc + 3ac^2 + 8b^3 - 12b^2c + 6bc^2 - c^3.$$

NOTE. A full discussion of the Binomial Theorem for Positive Integral Index is given in Chapter XXXVII.

EXAMPLES XX. e.

Expand the following expressions:

1. $(x + y)^5$.
2. $(a - b)^5$.
3. $(2a + b)^6$.
4. $(3x + 2y)^5$.
5. $(2x - 3b)^5$.
6. $(c^2 + d^3)^6$.
7. $(2ab - c^2)^5$.
8. $(3a^2b^2 - 2cd^3)^5$.
9. $(\frac{1}{2}x^2 + z^3)^5$.
10. $\left(\frac{a^2b}{3} - \frac{c^2}{4}\right)^5$.
11. $\left(\frac{2abc}{3} - \frac{2d^2}{5}\right)^4$.
12. $\left(\frac{2a}{x} - \frac{b^2}{z}\right)^5$.
13. $(a + 2b + 3c)^3$.
14. $(a - b + c)^3$.
15. $(2a - \frac{b}{2} - c^2)^3$.
16. $(a + b - 2c)^4$.
17. $(a + b + c - d)^3$.
18. $(a + 2b + c - 2d)^3$.
19. $(a - 2b + c - d)^4$.
20. Find the middle term of $(a + 1)^{10}$.
21. Find the two middle terms of $(x - y)^{11}$.
22. Find the term independent of a in $\left(\frac{a^4}{2} - \frac{2}{a^3}\right)^7$.

CHAPTER XXI.

Evolution.

195. The root of any proposed expression is that quantity which being repeated as a factor the requisite number of times produces the given expression. (Art. 14.)

The operation of finding the root is called **Evolution**: it is the inverse of Involution.

196. By the **Rule of Signs** we see that

(1) any **even root** of a *positive* quantity may be either *positive* or *negative*:

(2) no **negative** quantity can have an *even* root:

(3) every **odd root** of a quantity has the same sign as the quantity itself.

Note. It is especially worthy of notice that every positive quantity has two square roots equal in magnitude, but opposite in sign.

Ex. $$\sqrt{(9a^2x^6)} = \pm 3ax^3.$$

In the present chapter, however, we shall confine our attention to the positive root.

EVOLUTION OF MONOMIALS.

197. From a consideration of the following examples we will be able to deduce a general rule for extracting any proposed root of a monomial.

Examples. (1) $\sqrt{(a^6b^4)} = a^3b^2$ because $(a^3b^2)^2 = a^6b^4$.

(2) $\sqrt[3]{(-x^9)} = -x^3$ because $(-x^3)^3 = -x^9$.

(3) $\sqrt[5]{(c^{20})} = c^4$ because $(c^4)^5 = c^{20}$.

(4) $\sqrt[4]{(81x^{12})} = 3x^3$ because $(3x^3)^4 = 81x^{12}$.

Rule. (1) *Find the root of the coefficient by Arithmetic, and prefix the proper sign found by Art. 42.*

(2) *Divide the exponent of every factor of the expression by the index of the proposed root.*

EXAMPLES. (1) $\sqrt[3]{(-64x^6)} = -4x^2$.

(2) $\sqrt[4]{(16a^8)} = 2a^2$.

(3) $\sqrt{\left(\dfrac{81x^{10}}{25c^4}\right)} = \dfrac{9x^5}{5c^2}$.

It will be seen that in the last case we operate separately upon the numerator and the denominator.

EXAMPLES XXI. a.

Write the square root of each of the following expressions:

1. $4a^2b^4$.
2. $9x^6y^2$.
3. $25x^4y^6$.
4. $16a^4b^2c^6$.
5. $81a^6b^8$.
6. $100x^8$.
7. $a^{20}b^{16}c^4$.
8. $a^8b^2c^{12}$.
9. $\dfrac{324x^{12}}{169y^6}$.
10. $\dfrac{81a^{18}}{36b^{12}}$.
11. $\dfrac{256x^2y^4}{289p^{14}}$.
12. $\dfrac{400a^4b^{20}}{81x^{16}y^{16}}$.

Write the cube root of each of the following expressions:

13. $27a^6b^3c^3$.
14. $-8a^{12}b^9$.
15. $64x^6y^3z^{12}$.
16. $-343a^{12}b^{15}$.
17. $-\dfrac{x^{12}y^9}{125}$.
18. $\dfrac{8x^9}{729y^{15}}$.
19. $\dfrac{125a^3b^6}{216x^6y^9}$.
20. $-\dfrac{27x^{27}}{64y^{63}}$.
21. $-\dfrac{343x^{15}}{512z^{21}}$.

Write the value of each of the following expressions:

22. $\sqrt[6]{(729a^{18}b^6)}$.
23. $\sqrt[8]{(256a^8x^{64})}$.
24. $\sqrt[5]{(-x^{10}y^{15})}$.
25. $\sqrt[7]{\dfrac{128}{a^{63}b^{56}}}$.
26. $\sqrt[10]{\dfrac{a^{30}x^{50}}{b^{100}}}$.
27. $\sqrt[9]{\dfrac{a^{18}}{b^{27}c^{36}}}$.
28. $\sqrt[11]{\dfrac{x^{11}y^{44}}{z^{121}}}$.

EVOLUTION OF MULTINOMIALS.

198. The Square Root of Any Multinomial. Since the square of $a+b$ is $a^2+2ab+b^2$, we have to discover a process by which a and b, the terms of the root, can be found when $a^2+2ab+b^2$ is given.

The first term, a, is the square root of a^2.

Arrange the terms according to powers of one letter a. The first term is a^2, and its square root is a. Set this down as the first term of the required root. Subtract a^2 from the given expression and the remainder is $2ab + b^2$ or $(2a + b) \times b$.

Thus, b, the second term of the root, will be the quotient when the remainder is divided by $2a + b$.

This divisor consists of two terms:

1. The double of a, the term of the root already found.
2. b, the new term itself.

The work may be arranged as follows:

$$\begin{array}{r|l} & a^2 + 2ab + b^2\,(a + b \\ & a^2 \\ \hline 2a + b & 2ab + b^2 \\ & 2ab + b^2 \\ \hline \end{array}$$

Ex. **1.** Find the square root of $9x^2 - 42xy + 49y^2$.

$$\begin{array}{r|l} & 9x^2 - 42xy + 49y^2\,(3x - 7y \\ & 9x^2 \\ \hline 6x - 7y & -42xy + 49y^2 \\ & -42xy + 49y^2 \\ \hline \end{array}$$

Explanation. The square root of $9x^2$ is $3x$, and this is the first term of the root.

By doubling this we obtain $6x$, which is the first term of the divisor. Divide $-42xy$, the first term of the remainder, by $6x$ and we get $-7y$, the new term in the root, which has to be annexed both to the root and divisor. Next multiply the complete divisor by $-7y$ and subtract the result from the first remainder. There is now no remainder and the root has been found.

The process can be extended so as to find the square root of any multinomial. The first two terms of the root will be obtained as before. When we have brought down *the second remainder*, the first part of the new divisor is obtained by doubling the terms of the root already found. We then divide the first term of the remainder by the first term of the new divisor, and set down the result as the next term in the root and in the divisor. We next multiply the complete divisor by the last term of the root and subtract the product from the last remainder. If there is now no remainder the root has been found; if there is a remainder we continue the process.

Ex. **2.** Find the square root of

$$25x^2a^2 - 12xa^3 + 16x^4 + 4a^4 - 24x^3a.$$

Rearrange in descending powers of x.

$$
\begin{array}{ll}
 & 16x^4 - 24x^3a + 25x^2a^2 - 12xa^3 + 4a^4\,(4x^2 - 3xa + 2a^2 \\
 & \underline{16x^4} \\
8x^2 - 3xa & \quad -24x^3a + 25x^2a^2 \\
 & \quad \underline{-24x^3a + 9x^2a^2} \\
8x^2 - 6xa + 2a^2 & \qquad\qquad 16x^2a^2 - 12xa^3 + 4a^4 \\
 & \qquad\qquad \underline{16x^2a^2 - 12xa^3 + 4a^4}
\end{array}
$$

EXPLANATION. When we have obtained two terms in the root, $4x^2 - 3xa$, we have a remainder

$$16x^2a^2 - 12xa^3 + 4a^4.$$

Double the terms of the root already found and place the result, $8x^2 - 6xa$, as the first part of the divisor. Divide $16x^2a^2$, the first term of the remainder, by $8x^2$, the first term of the divisor; we get $+2a^2$ which we annex both to the root and divisor. Now multiply the complete divisor by $2a^2$ and subtract. There is no remainder and the root is found.

EXAMPLES XXI. b.

Find the square root of each of the following expressions:

1. $x^2 - 10xy + 25y^2$.

2. $4x^2 - 12xy + 9y^2$.

3. $81x^2 + 18xy + y^2$.

4. $25x^2 - 30xy + 9y^2$.

5. $a^4 - 2a^3 + 3a^2 - 2a + 1$.

6. $4x^4 - 12x^3 + 29x^2 - 30x + 25$.

7. $9x^4 - 12x^3 - 2x^2 + 4x + 1$.

8. $x^4 - 4x^3 + 6x^2 - 4x + 1$.

9. $4a^4 + 4a^3 - 7a^2 - 4a + 4$.

10. $1 - 10x + 27x^2 - 10x^3 + x^4$.

11. $4x^2 + 9y^2 + 25z^2 + 12xy - 30yz - 20xz$.

12. $16x^6 + 16x^7 - 4x^8 - 4x^9 + x^{10}$.

13. $x^6 - 22x^4 + 34x^3 + 121x^2 - 374x + 289$.

14. $25x^4 - 30ax^3 + 49a^2x^2 - 24a^3x + 16a^4$.

15. $4x^4 + 4x^2y^2 - 12x^2z^2 + y^4 - 6y^2z^2 + 9z^4$.

16. $6ab^2c - 4a^2bc + a^2b^2 + 4a^2c^2 + 9b^2c^2 - 12abc^2$.

17. $-6b^2c^2 + 9c^4 + b^4 - 12c^2a^2 + 4a^4 + 4a^2b^2$.

18. $4x^4 + 9y^4 + 13x^2y^2 - 6xy^3 - 4x^3y$.

19. $1 - 4x + 10x^2 - 20x^3 + 25x^4 - 24x^5 + 16x^6$.

20. $6acx^5 + 4b^2x^4 + a^2x^{10} + 9c^2 - 12bcx^2 - 4abx^7$.

199. When the expression whose root is required contains fractional terms, we may proceed as before, the fractional part of the work being performed by the rules explained in Chapter XIII.

200. There is one important point to be observed when an expression contains powers of a certain letter and also powers of its reciprocal. Thus in the expression

$$2x + \frac{1}{x^2} + 4 + x^3 + \frac{5}{x} + 7x^2 + \frac{8}{x^3},$$

the order of *descending* powers is

$$x^3 + 7x^2 + 2x + 4 + \frac{5}{x} + \frac{1}{x^2} + \frac{8}{x^3};$$

and the numerical quantity 4 stands between x and $\frac{1}{x}$.

The reason for this arrangement will appear in Chapter XXII.

Ex. Find the square root of $24 + \frac{16y^2}{x^2} - \frac{8x}{y} + \frac{x^2}{y^2} - \frac{32y}{x}$.

Arrange the expression in descending powers of y.

$$\begin{array}{l|l}
 & \frac{16y^2}{x^2} - \frac{32y}{x} + 24 - \frac{8x}{y} + \frac{x^2}{y^2} \left(\frac{4y}{x} - 4 + \frac{x}{y} \right. \\
 & \underline{\frac{16y^2}{x^2}} \\
\frac{8y}{x} - 4 & \quad - \frac{32y}{x} + 24 \\
 & \quad \underline{- \frac{32y}{x} + 16} \\
\frac{8y}{x} - 8 + \frac{x}{y} & \qquad\qquad 8 - \frac{8x}{y} + \frac{x^2}{y^2} \\
 & \qquad\qquad \underline{8 - \frac{8x}{y} + \frac{x^2}{y^2}}
\end{array}$$

Here the second term in the root, -4, arises from division of $-\frac{32y}{x}$ by $\frac{8y}{x}$, and the third term, $\frac{x}{y}$, arises from division of 8 by $\frac{8y}{x}$; thus $8 \div \frac{8y}{x} = 8 \times \frac{x}{8y} = \frac{x}{y}$.

EXAMPLES XXI. c.

Find the square root of each of the following expressions

1. $\frac{x^2}{25}+\frac{2xy}{5}+y^2$.

2. $\frac{x^2}{y^2}+\frac{10x}{y}+25$.

3. $\frac{x^2}{4y^2}-\frac{2x}{y}+4$.

4. $\frac{x^2}{y^2}-\frac{2ax}{by}+\frac{a^2}{b^2}$.

5. $\frac{64x^2}{9y^2}+\frac{32x}{3y}+4$.

6. $\frac{9x^2}{25}-2+\frac{25}{9x^2}$.

7. $\frac{a^4}{64}+\frac{a^3}{8}-a+1$.

8. $x^4+2x^3-x+\frac{1}{4}$.

9. $-3a^3+\frac{25}{9}+a^4-5a+\frac{67}{12}a^2$.

10. $x^4-2x+\frac{1}{9}+\frac{29}{3}x^2-6x^3$.

11. $\frac{a^4}{4}+\frac{a^3}{x}+\frac{a^2}{x^2}-ax-2+\frac{x^2}{a^2}$.

12. $x^4-2x^3+\frac{3x^2}{2}-\frac{x}{2}+\frac{1}{16}$.

13. $\frac{x^4}{4}+4x^2+\frac{ax^2}{3}+\frac{a^2}{9}-2x^3-\frac{4ax}{3}$.

14. $\frac{9a^2}{x^2}-\frac{6a}{5x}+\frac{101}{25}-\frac{4x}{15a}+\frac{4x^2}{9a^2}$.

15. $16m^4+\frac{16}{3}m^2n+8m^2+\frac{4}{9}n^2+\frac{4}{3}n+1$.

16. $4x^4+32x^2+96+\frac{64}{x^4}+\frac{128}{x^2}$.

201. The Cube Root of Any Multinomial. Since the cube of $a+b$ is $a^3+3a^2b+3ab^2+b^3$, we have to discover a process by which a and b, the terms of the root, can be found when $a^3+3a^2b+3ab^2+b^3$ is given.

The first term a is the cube root of a^3.

Arrange the terms according to powers of one letter a; then the first term is a^3, and its cube root a. Set this down as the first term of the required root. Subtract a^3 from the given expression, and the remainder is

$$3a^2b+3ab^2+b^3 \text{ or } (3a^2+3ab+b^2)\times b.$$

Now the first term of the remainder is the product of $3a^2$ and b. Thus to obtain b we divide the first term of the remainder by three times the square of the term already found.

Having found b we can complete the divisor, which consists of the following three terms:

1. Three times the square of a, the term of the root already found.

2. Three times the product of this first term a and the new term b.

3. The square of b.

The work may be arranged as follows:

$$
\begin{array}{lllr|l}
 & & & & a^3 + 3a^2b + 3ab^2 + b^3(a + b \\
 & & & & a^3 \\ \hline
3(a)^2 & = 3a^2 & & & 3a^2b + 3ab^2 + b^3 \\
3 \times a \times b = & & +3ab & & \\
(b)^2 & = & & +b^2 & \\ \cline{2-4}
 & 3a^2 + 3ab + b^2 & & & 3a^2b + 3ab^2 + b^3
\end{array}
$$

Ex. 1. Find the cube root of $8x^3 - 36x^2y + 54xy^2 - 27y^3$.

$$
\begin{array}{llll|l}
 & & & & 8x^3 - 36x^2y + 54xy^2 - 27y^3(2x - 3y \\
 & & & & 8x^3 \\ \hline
3(2x)^2 & = 12x^2 & & & -36x^2y + 54xy^2 - 27y^3 \\
3 \times 2x \times (-3y) = & & -18xy & & \\
(-3y)^2 = & & & +9y^2 & \\ \cline{2-4}
 & 12x^2 - 18xy + 9y^2 & & & -36x^2y + 54xy^2 - 27y^3
\end{array}
$$

Explanation. The cube root of $8x^3$ is $2x$, and this is the first term of the root.

By taking three times the square of this first term we obtain $12x^2$, which is the first term of the divisor, and is called the "trial divisor." Divide $-36x^2y$, the first term of the remainder, by $12x^2$ and we get $-3y$, the new term in the root. To complete the divisor, we first annex to the trial divisor three times the product of $2x$, the part of the root already found, and $-3y$, the new term of the root: this is $-18xy$. We then annex the square of $-3y$, the new term, and the divisor is complete. We next multiply this divisor by the new term, and subtract the result from the first remainder. There is now no remainder and the root has been found.

The process can be extended so as to find the cube root of any multinomial. The first two terms of the root will be obtained as before. When we have brought down *the second remainder*, we form the trial divisor by taking three times the square of the two terms of the root already found, and proceed as is shown in the following example.

Ex. **2.** Find the cube root of $27 + 108x + 90x^2 - 80x^3 - 60x^4 + 48x^5 - 8x^6$.

$$
\begin{array}{lr|l}
 & & 27 + 108x + 90x^2 - 80x^3 - 60x^4 + 48x^5 - 8x^6 \,(3 + 4x - 2x^2 \\
 & & \underline{27} \\
3 \times (3)^2 = 27 & & 108x + 90x^2 - 80x^3 \\
3 \times 3 \times 4x = \quad +36x & & \\
(4x)^2 = \quad \underline{+16x^2} & & \\
27 + 36x + 16x^2 & & \underline{108x + 144x^2 + 64x^3} \\
3 \times (3 + 4x)^2 = 27 + 72x + 48x^2 & & -54x^2 - 144x^3 - 60x^4 + 48x^5 - 8x^6 \\
3 \times (3 + 4x) \times (-2x^2) = \quad -18x^2 - 24x^3 & & \\
(-2x^2)^2 = \quad \underline{+4x^4} & & \\
27 + 72x + 30x^2 - 24x^3 + 4x^4 & & \underline{-54x^2 - 144x^3 - 60x^4 + 48x^5 - 8x^6}
\end{array}
$$

Explanation. When we have obtained two terms in the root, $3 + 4x$, we have a remainder

$$-54x^2 - 144x^3 - 60x^4 + 48x^5 - 8x^6.$$

Take 3 times the square of the root already found and place the result, $27 + 72x + 48x^2$, as the first part of the new divisor. Divide $-54x^2$, the first term of the remainder, by 27, the first term of the divisor; this gives a new term of the root $-2x^2$. To complete the divisor we take 3 times the product of $(3 + 4x)$ and $-2x^2$, and also the square of $-2x^2$. Now multiply the complete divisor by $-2x^2$ and subtract; there is no remainder, and the root is found.

EXAMPLES XXI. d.

Find the cube root of each of the following expressions:

1. a^3+3a^2+3a+1.
2. $a^3x^3-3a^2x^2y^2+3axy^4-y^6$.
3. $64a^3-144a^2b+108ab^2-27b^3$.
4. $1+3x+6x^2+7x^3+6x^4+3x^5+x^6$.
5. $1-6x+21x^2-44x^3+63x^4-54x^5+27x^6$.
6. $a^3+6a^2b-3a^2c+12ab^2-12abc+3ac^2+8b^3-12b^2c+6bc^2-c^3$.
7. $8a^6-36a^5+66a^4-63a^3+33a^2-9a+1$.
8. $8x^6+12x^5-30x^4-35x^3+45x^2+27x-27$.
9. $27x^6-54x^5a+117x^4a^2-116x^3a^3+117x^2a^4-54xa^5+27a^6$.
10. $27x^6-27x^5-18x^4+17x^3+6x^2-3x-1$.
11. $24x^4y^2+96x^2y^4-6x^5y+x^6-96xy^5+64y^6-56x^3y^3$.
12. $216+342x^2+171x^4+27x^6-27x^5-109x^3-108x$.

202. We add some examples of cube root where fractional terms occur in the given expressions.

Ex. Find the cube root of $54-27x^3+\dfrac{8}{x^6}-\dfrac{36}{x^3}$.

Arrange the expression in *ascending* powers of x.

$$\begin{array}{lr|l}
 & & \dfrac{8}{x^6}-\dfrac{36}{x^3}+54-27x^3\left(\dfrac{2}{x^2}-3x\right. \\
 & & \dfrac{8}{x^6} \\ \hline
3\times\left(\dfrac{2}{x^2}\right)^2 & =\dfrac{12}{x^4} & -\dfrac{36}{x^3}+54-27x^3 \\
3\times\dfrac{2}{x^2}\times(-3x)= & -\dfrac{18}{x} & \\
(-3x)^2 & =\qquad +9x^2 & \\ \cline{2-2}
 & \dfrac{12}{x^4}-\dfrac{18}{x}+9x^2 & -\dfrac{36}{x^3}+54-27x^3 \\ \hline
\end{array}$$

EXAMPLES XXI. e.

Find the cube root of each of the following expressions:

1. $\dfrac{x^3}{8}-\dfrac{3x^2}{4}+\dfrac{3x}{2}-1$.
2. $8x^3-4x^2y^2+\frac{2}{3}xy^4-\dfrac{y^6}{27}$.
3. $\dfrac{27x^3}{64y^3}-\dfrac{27x^2}{8y^2}+\dfrac{9x}{y}-8$.
4. $\dfrac{x^6}{y^3}-6x^4+12x^2y^3-8y^6$.

5. $\frac{x^3}{y^3}+\frac{6x^2}{y^2}+\frac{9x}{y}-4-\frac{9y}{x}+\frac{6y^2}{x^2}-\frac{y^3}{x^3}.$

6. $\frac{x^3}{27}-\frac{x^2}{3}+2x-7+\frac{18}{x}-\frac{27}{x^2}+\frac{27}{x^3}.$

7. $\frac{x^3}{a^3}-\frac{12x^2}{a^2}+\frac{54x}{a}-112+\frac{108a}{x}-\frac{48a^2}{x^2}+\frac{8a^3}{x^3}.$

8. $\frac{64a^3}{x^3}-\frac{192a^2}{x^2}+\frac{240a}{x}-160+\frac{60x}{a}-\frac{12x^2}{a^2}+\frac{x^3}{a^3}.$

9. $\frac{6b}{a}+\frac{6a}{b}-7+\frac{a^3}{b^3}-\frac{3a^2}{b^2}-\frac{3b^2}{a^2}+\frac{b^3}{a^3}.$

10. $\frac{60x^4}{y^4}-\frac{80x^3}{y^3}-\frac{90x^2}{y^2}+\frac{8x^6}{y^6}+\frac{108x}{y}-27+\frac{48x^5}{y^5}.$

203. Some Higher Roots. The *fourth* root of an expression is obtained by extracting the square root of the square root of the expression.

Similarly by successive applications of the rule for finding the square root, we may find the *eighth, sixteenth* ⋯ root. The *sixth* root of an expression is found by taking the cube root of the square root, or the square root of the cube root.

Similarly by combining the two processes for extraction of cube and square roots, other higher roots may be obtained.

Ex. **1.** Find the fourth root of

$$81x^4-216x^3y+216x^2y^2-96xy^3+16y^4.$$

Extracting the square root by the rule we obtain $9x^2-12xy+4y^2$; and *by inspection*, the square root of this is $3x-2y$, which is the required fourth root.

Ex. **2.** Find the sixth root of

$$\left(x^3-\frac{1}{x^3}\right)^2-6\left(x-\frac{1}{x}\right)\left(x^3-\frac{1}{x^3}\right)+9\left(x-\frac{1}{x}\right)^2.$$

By inspection, the square root of this is

$$\left(x^3-\frac{1}{x^3}\right)-3\left(x-\frac{1}{x}\right),$$

which may be written $\quad x^3-3x+\frac{3}{x}-\frac{1}{x^3};$

and the cube root of this is $\quad x-\frac{1}{x},$

which is the required sixth root.

We conclude the subject of higher roots by giving a rule, which depends upon the Binomial Theorem, for finding the **nth root** of any multinomial.

(1) *Arrange the terms according to the descending powers of some letter.*

(2) *Take the nth root of the first term, and this will be the first term of the root.*

(3) *When* ANY NUMBER OF TERMS *of the root have been found, subtract from the given multinomial the nth power of the part of the root already found, and divide the first term of the remainder by n times the* $(n-1)$*th power of the* FIRST TERM *of the root, and this will be the next term of the root.*

204. When an expression is not an exact square or cube, we may perform the process of evolution, and obtain as many terms of the root as we please.

Ex. To find four terms of the square root of $1+2x-2x^2$.

$$\begin{array}{ll}
 & 1+2x-2x^2\,(1+x-\frac{3}{2}x^2+\frac{3}{2}x^3 \\
 & 1 \\
2+x & \quad 2x-2x^2 \\
 & \quad 2x+\ \ x^2 \\
2+2x-\frac{3}{2}x^2 & \qquad -3x^2 \\
 & \qquad -3x^2-3x^3+\frac{9}{4}x^4 \\
2+2x-3x^2+\frac{3}{2}x^3 & \qquad\qquad 3x^3-\frac{9}{4}x^4 \\
 & \qquad\qquad 3x^3+3x^4-\frac{9}{2}x^5+\frac{9}{4}x^6 \\
 & \qquad\qquad\quad -\frac{21}{4}x^4+\frac{9}{2}x^5-\frac{9}{4}x^6.
\end{array}$$

Thus the required result is $1+x-\frac{3}{2}x^2+\frac{3}{2}x^3$.

EXAMPLES XXI. f.

Find the fourth roots of the following expressions:

1. $x^4-28x^3+294x^2-1372x+2401$.
2. $16-\frac{32}{m}+\frac{24}{m^2}-\frac{8}{m^3}+\frac{1}{m^4}$.
3. $a^4+8a^3x+16x^4+32ax^3+24a^2x^2$.
4. $1+4x+2x^2-8x^3-5x^4+8x^5+2x^6-4x^7+x^8$.
5. $1+8x+20x^2+8x^3-26x^4-8x^5+20x^6-8x^7+x^8$.

Find the sixth roots of the following expressions:

6. $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6.$

7. $x^6 - 12ax^5 + 240a^4x^2 - 192a^5x + 60a^2x^4 - 160a^3x^3 + 64a^6.$

8. $a^6 - 18a^5x + 135a^4x^2 - 540a^3x^3 + 1215a^2x^4 - 1458ax^5 + 729x^6.$

Find the eighth roots of the following expressions:

9. $x^8 - 8x^7y + 28x^6y^2 - 56x^5y^3 + 70x^4y^4 - 56x^3y^5 + 28x^2y^6 - 8xy^7 + y^8.$

10. $\{x^4 + 2(p-1)x^3 + (p^2 - 2p - 1)x^2 - 2(p-1)x + 1\}^4.$

Find to four terms the square root of

11. $a^2 - x.$ **12.** $x^2 + a^2.$ **13.** $a^4 - 3x^2.$ **14.** $9a^2 + 12ax.$

Find to three terms the cube root of

15. $1 - 6x + 21x^2.$ **16.** $27x^6 - 27x^5 - 18x^4.$ **17.** $64 - 48x + 9x^2.$

18. Find the fifth root of

$$a^{10} - 10a^9 + 50a^8 - 160a^7 + 360a^6 - 592a^5 + 720a^4 - 640a^3 + 400a^2 - 160a + 32.$$

19. $a^{10} - 5a^9 + 20a^8 - 50a^7 + 105a^6 - 161a^5 + 210a^4 - 200a^3 + 160a^2 - 80a + 32.$

20. $a^{10} + 5a^9 + 5a^8 - 10a^7 - 15a^6 + 11a^5 + 15a^4 - 10a^3 - 5a^2 + 5a - 1.$

205. Square and Cube Root of Numbers. Before leaving the subject of Evolution it may be useful to remark that the ordinary rules for extracting square and cube roots in Arithmetic are based upon the algebraic methods we have explained in the present chapter.

Ex. 1. Find the square root of 5329.

Since 5329 lies between 4900 and 6400, that is between $(70)^2$ and $(80)^2$, its square root consists of two figures and lies between 70 and 80. Hence, corresponding to a, the first term of the root in the algebraic process of Art. 198, we here have 70.

The analogy between the algebraic and arithmetical methods will be seen by comparing the cases we give below.

$$\begin{array}{l|l}
 & a^2 + 2ab + b^2(a + b \\
 & \underline{a^2} \\
2a + b & 2ab + b^2 \\
 & \underline{2ab + b^2}
\end{array}
\qquad\qquad
\begin{array}{r|l}
 & 5329(70 + 3 = 73. \\
 & \underline{4900} \\
140 + 3 = 143 & 429 \\
 & \underline{429}
\end{array}$$

Ex. 2. Find the square root of 53824.

Here 53824 lies between 40000 and 90000, that is between $(200)^2$ and $(300)^2$.

$$\begin{array}{llr|l}
 & & & \quad a \quad\;\; b \quad\;\; c \\
 & & & 53824\,(200+30+2=232 \\
 & & & 40000 \\
\hline
2a+b\ \dots & 400+30=430 & & 13824 \\
 & & & 12900 \\
\hline
2(a+b)+c\ \dots & 460+\ \ 2=462 & & 924 \\
 & & & 924 \\
\end{array}$$

Ex. 3. Find the cube root of 614125.

Since 614125 lies between 512000 and 729000, that is between $(80)^3$ and $(90)^3$, therefore its cube root consists of two figures and lies between 80 and 90.

$$\begin{array}{lrr|l}
 & & & \quad a+b \\
 & & & 614125\,(80+5=85. \\
 & & & 512000 \\
\hline
3a^2=3\times(80)^2 & = & 19200 & 102125 \\
3\times a\times b=3\times 80\times 5 & = & 1200 & \\
b^2= \qquad 5\times 5 & = & 25 & \\
\hline
 & & 20425 & 102125 \\
\end{array}$$

206. We shall now show that in extracting either the square or the cube root of any number, when a certain number of figures have been obtained by the common rule, that number may be nearly doubled by ordinary division.

207. If the square root of a number consists of $2n+1$ figures, when the first $n+1$ of these have been obtained by the ordinary method, the remaining n may be obtained by division.

Let N denote the given number; a the part of the square root already found, that is the first $n+1$ figures found by the common rule, with n ciphers annexed; x the remaining part of the root.

Then
$$\sqrt{N}=a+x;$$
$$\therefore\ N=a^2+2ax+x^2;$$
$$\therefore\ \frac{N-a^2}{2a}=x+\frac{x^2}{2a} \quad \dots\dots\dots \quad (1).$$

Now $N - a^2$ is the remainder after $n+1$ figures of the root, represented by a, have been found; and $2a$ is the divisor at the same stage of the work. We see from (1) that $N - a^2$ divided by $2a$ gives x, the rest of the quotient required, increased by $\frac{x^2}{2a}$. We shall show that $\frac{x^2}{2a}$ is a *proper fraction*, so that by neglecting the remainder arising from the division, we obtain x, the rest of the root.

For x contains n figures, and therefore x^2 contains $2n$ figures at most; also a is a number of $2n+1$ figures (the last n of which are ciphers) and thus $2a$ contains $2n+1$ figures at least; and therefore $\frac{x^2}{2a}$ is a proper fraction.

From the above investigation, by putting $n=1$, we see that *two* at least of the figures of a square root must have been obtained in order that the method of division, used to obtain the next figure of the square root, may give that figure correctly.

Ex. Find the square root of 290 to five places of decimals.

$$\begin{array}{r|l}
 & \dot{2}9\dot{0}\,(17.02 \\
 & 1 \\ \hline
27 & 190 \\
 & 189 \\ \hline
3402 & 10000 \\
 & 6804 \\ \hline
 & 3196
\end{array}$$

Here we have obtained four figures in the square root by the ordinary method. Three more may be obtained by division only, using 2×1702, that is 3404, for divisor, and 3196 as remainder. Thus

$$\begin{array}{r}
3404)\,31960\,(938 \\
\underline{30636} \\
13240 \\
\underline{10212} \\
30280 \\
\underline{27232} \\
3048
\end{array}$$

And therefore to five places of decimals $\sqrt{290} = 17.02938$.

It will be noticed that in obtaining the second figure of the root, the division of 190 by 20 gives 9 for the next figure; this is too great, and the figure 7 has to be obtained tentatively.

208. If the cube root of a number consists of $2n+2$ figures, when the first $n+2$ of these have been obtained by the ordinary method, the remaining n may be obtained by division.

Let N denote the given number; a the part of the cube root already found, that is, the first $n+2$ figures found by the common rule, with n ciphers annexed; x the remaining part of the root.

Then $$\sqrt[3]{N} = a + x;$$
$$N = a^3 + 3a^2x + 3ax^2 + x^3;$$
$$\therefore \frac{N-a^3}{3a^2} = x + \frac{x^2}{a} + \frac{x^3}{3a^2} \quad \ldots \ldots \quad (1).$$

Now $N - a^3$ is the remainder after $n+2$ figures of the root, represented by a, have been found; and $3a^2$ is the divisor at the same stage of the work. We see from (1) that $N - a^3$ divided by $3a^2$ gives x, the rest of the quotient required, increased by $\frac{x^2}{a} + \frac{x^3}{3a^2}$. We shall show that this expression is a *proper fraction*, so that by neglecting the remainder arising from the division, we obtain x, the rest of the root.

By supposition, x is less than 10^n, and a is greater than 10^{2n+1}; therefore $\frac{x^2}{a}$ is less than $\frac{10^{2n}}{10^{2n+1}}$; that is, less than $\frac{1}{10}$; and $\frac{x^3}{3a^2}$ is less than $\frac{10^{3n}}{3 \times 10^{4n+2}}$; that is, less than $\frac{1}{3 \times 10^{n+1}}$; hence $\frac{x^2}{a} + \frac{x^3}{3a^2}$ is less than $\frac{1}{10} + \frac{1}{3 \times 10^{n+1}}$, and is therefore a proper fraction.

CHAPTER XXII.

The Theory of Indices.

209. Hitherto all the definitions and rules with regard to indices have been based upon the supposition that they were positive integers; for instance,

(1) $a^{14} = a \cdot a \cdot a \cdots$ to fourteen factors.

(2) $a^{14} \times a^3 = a^{14+3} = a^{17}$.

(3) $a^{14} \div a^3 = a^{14-3} = a^{11}$.

(4) $(a^{14})^3 = a^{14 \times 3} = a^{42}$.

The object of this chapter is twofold; first, to give *general* proofs which shall establish the laws of combination in the case of positive integral indices; secondly, to explain how, in strict accordance with these laws, intelligible meanings may be given to symbols whose indices are fractional, zero, or negative.

We shall begin by proving, directly from the definition of a positive integral index, three important propositions.

210. Definition. When m is a *positive integer*, a^m stands for the product of m factors each equal to a.

211. Prop. I. **To prove that $a^m \times a^n = a^{m+n}$, when m and n are positive integers.**

By definition, $a^m = a \cdot a \cdot a \cdots$ to m factors;

$a^n = a \cdot a \cdot a \cdots$ to n factors;

$\therefore a^m \times a^n = (a \cdot a \cdot a \cdots$ to m factors$) \times (a \cdot a \cdot a \cdots$ to n factors$)$

$= a \cdot a \cdot a \cdots$ to $m + n$ factors

$= a^{m+n}$, by definition.

Cor. If p is also a positive integer, then

$$a^m \times a^n \times a^p = a^{m+n+p};$$

and so for any number of factors.

212. Prop. II. **To prove that $a^m \div a^n = a^{m-n}$, when m and n are positive integers, and m is greater than n.**

$$a^m \div a^n = \frac{a^m}{a^n} = \frac{a \cdot a \cdot a \cdots \text{ to } m \text{ factors}}{a \cdot a \cdot a \cdots \text{ to } n \text{ factors}}$$

$$= a \cdot a \cdot a \cdots \text{ to } m-n \text{ factors}$$

$$= a^{m-n}.$$

213. Prop. III. **To prove that $(a^m)^n = a^{mn}$, when m and n are positive integers.**

$$(a^m)^n = a^m \cdot a^m \cdot a^m \cdots \text{ to } n \text{ factors}$$

$$= (a \cdot a \cdot a \cdots \text{ to } m \text{ factors})\,(a \cdot a \cdot a \cdots \text{ to } m \text{ factors}) \cdots$$

the bracket being repeated n times,

$$= a \cdot a \cdot a \cdots \text{ to } mn \text{ factors}$$

$$= a^{mn}.$$

214. These are the fundamental laws of combination of indices, and they are proved directly from a definition which is intelligible only on the supposition that the indices are *positive* and *integral*.

But it is found convenient to use fractional and negative indices, such as $a^{\frac{4}{5}}$, a^{-7}; or, more generally, $a^{\frac{p}{q}}$, a^{-n}; and these have at present no intelligible meaning. For the definition of a^m [Art. 210], upon which we based the three propositions just proved, is no longer applicable when m is *fractional* or *negative*.

Now it is important that all indices, whether positive or negative, integral or fractional, should be governed by the same laws. We therefore determine meanings for symbols such as $a^{\frac{p}{q}}$, a^{-n}, in the following way: we assume that they conform to the fundamental law, $a^m \times a^n = a^{m+n}$, and accept the meaning to which this assumption leads us. It will be found that the symbols so interpreted will also obey the other laws enunciated in Props. II. and III.

215. To find a meaning for $a^{\frac{p}{q}}$, p and q being positive integers.

Since $a^m \times a^n = a^{m+n}$ is to be true for *all* values of m and n, by replacing each of the indices m and n by $\frac{p}{q}$, we have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}} = a^{\frac{2p}{q}}.$$

Similarly, $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{2p}{q}} \times a^{\frac{p}{q}} = a^{\frac{2p}{q}+\frac{p}{q}} = a^{\frac{3p}{q}}.$

Proceeding in this way for 4, 5, $\cdots q$ factors, we have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} \cdots \text{ to } q \text{ factors} = a^{\frac{qp}{q}};$$

that is,
$$(a^{\frac{p}{q}})^q = a^p.$$

Therefore, by taking the qth root,

$$a^{\frac{p}{q}} = \sqrt[q]{a^p},$$

or, in words, $a^{\frac{p}{q}}$ is equal to "*the qth root of a^p.*"

EXAMPLES. (1) $x^{\frac{5}{7}} = \sqrt[7]{x^5}$.

(2) $a^{\frac{1}{3}} = \sqrt[3]{a}$.

(3) $4^{\frac{3}{2}} = \sqrt{4^3} = \sqrt{64} = 8$.

(4) $a^{\frac{2}{3}} \times a^{\frac{5}{6}} = a^{\frac{2}{3}+\frac{5}{6}} = a^{\frac{3}{2}}$.

(5) $k^{\frac{a}{2}} \times k^{\frac{2}{3}} = k^{\frac{a}{2}+\frac{2}{3}} = k^{\frac{3a+4}{36}}$.

(6) $3a^{\frac{2}{3}}b^{\frac{1}{2}} \times 4a^{\frac{1}{6}}b^{\frac{5}{6}} = 12a^{\frac{2}{3}+\frac{1}{6}}b^{\frac{1}{2}+\frac{5}{6}} = 12a^{\frac{5}{6}}b^{\frac{4}{3}}$.

216. To find a meaning for a^0.

Since $a^m \times a^n = a^{m+n}$ is to be true for *all* values of m and n, by replacing the index m by 0, we have

$$a^0 \times a^n = a^{0+n} = a^n;$$

$$\therefore\ a^0 = \frac{a^n}{a^n} = 1.$$

Hence any quantity with zero index is equivalent to 1.

Ex. $x^{b-c} \times x^{c-b} = x^{b-c+c-b} = x^0 = 1$.

217. To find a meaning for a^{-n}.

Since $a^m \times a^n = a^{m+n}$ is to be true for *all* values of m and n, by replacing the index m by $-n$, we have

$$a^{-n} \times a^n = a^{-n+n} = a^0.$$

But $$a^0 = 1.$$

Hence $$a^{-n} = \frac{1}{a^n},$$

and $$a^n = \frac{1}{a^{-n}}.$$

From this it follows that *any* **factor** *may be transferred from the numerator to the denominator of an expression, or vice-versa, by merely changing the sign of the index.*

EXAMPLES. (1) $x^{-3} = \frac{1}{x^3}.$

(2) $\frac{1}{y^{-\frac{1}{2}}} = y^{\frac{1}{2}} = \sqrt{y}.$

(3) $27^{-\frac{2}{3}} = \frac{1}{27^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{(27)^2}} = \frac{1}{\sqrt[3]{3^6}} = \frac{1}{3^2} = \frac{1}{9}.$

218. To prove that $a^m \div a^n = a^{m-n}$ for all values of m and n.

$$a^m \div a^n = a^m \times \frac{1}{a^n} = a^m \times a^{-n}$$
$$= a^{m-n}, \text{ by the fundamental law.}$$

EXAMPLES. (1) $a^3 \div a^5 = a^{3-5} = a^{-2} = \frac{1}{a^2}.$

(2) $c \div c^{-\frac{8}{5}} = c^{1+\frac{8}{5}} = c^{\frac{13}{5}}.$

(3) $x^{a-b} \div x^{a-c} = x^{a-b-(a-c)} = x^{c-b}.$

219. The method of finding a meaning for a symbol, as explained in the preceding articles, deserves careful attention. The usual algebraic process is to make choice of symbols, give them meanings, and then prove the rules for their combination. Here the process is reversed; the symbols are given, and the law to which they are to conform, and from this the meanings of the symbols are determined.

220. The following examples will illustrate the different principles we have established.

EXAMPLES. (1) $\dfrac{3a^{-2}}{5x^{-1}y}=\dfrac{3x}{5a^2y}$.

(2) $\dfrac{2a^{\frac{1}{2}}\times a^{\frac{2}{3}}\times 6a^{-\frac{7}{3}}}{9a^{-\frac{5}{3}}\times a^{\frac{3}{2}}}=\dfrac{4}{3}a^{\frac{1}{2}+\frac{2}{3}-\frac{7}{3}+\frac{5}{3}-\frac{3}{2}}=\dfrac{4}{3}a^{-1}=\dfrac{4}{3a}$.

(3) $\dfrac{\sqrt{x^3}\times\sqrt[3]{y^2}}{\sqrt[6]{y^{-2}}\times\sqrt[4]{x^6}}=\dfrac{x^{\frac{3}{2}}\times y^{\frac{2}{3}}}{y^{-\frac{1}{3}}\times x^{\frac{3}{2}}}=x^{\frac{3}{2}-\frac{3}{2}}y^{\frac{2}{3}+\frac{1}{3}}=x^0y=y$.

(4) $2\sqrt{a}+\dfrac{3}{a^{-\frac{1}{2}}}+a^{\frac{5}{2}}=2a^{\frac{1}{2}}+3a^{\frac{1}{2}}+a^{\frac{5}{2}}$

$=5a^{\frac{1}{2}}+a^{\frac{5}{2}}=a^{\frac{1}{2}}(5+a^2)$.

EXAMPLES XXII. a.

Express with positive indices:

1. $2x^{-\frac{1}{4}}$.
2. $3a^{-\frac{2}{3}}$.
3. $4x^{-2}a^3$.
4. $3\div a^{-2}$.
5. $\dfrac{1}{4a^{-2}}$.
6. $\dfrac{1}{5x^{-\frac{1}{2}}}$.
7. $\dfrac{3a^{-3}x^2}{5y^2c^{-4}}$.
8. $\dfrac{x^ay^{-b}}{b^{-a}}$.
9. $2x^{\frac{1}{2}}\times 3x^{-1}$.
10. $1\div 2a^{-\frac{1}{2}}$.
11. $xy^2\times x^{-1}$.
12. $a^{-2}x^{-1}\div 3x$.
13. $\dfrac{1}{\sqrt{x^3}}$.
14. $\dfrac{1}{4\sqrt[5]{x^{-3}}}$.
15. $\dfrac{2}{\sqrt{y^{-3}}}$.
16. $\dfrac{\sqrt[4]{x^3}}{\sqrt{x^{-1}}}$.
17. $a^{-2}x^{-\frac{1}{2}}\div a^{-3}$.
18. $\sqrt[3]{a^{-1}}\div\sqrt[3]{a}$.
19. $\sqrt[5]{a^{-3}}\div\sqrt[5]{a^7}$.

Express with radical signs and positive indices:

20. $x^{\frac{3}{5}}$.
21. $a^{-\frac{1}{2}}$.
22. $5x^{-\frac{1}{2}}$.
23. $2a^{-\frac{1}{x}}$.
24. $\dfrac{1}{2a^{\frac{1}{3}}}$.
25. $\dfrac{2}{b^{-\frac{3}{4}}}$.
26. $\dfrac{c^{-\frac{1}{3}}}{2}$.
27. $\dfrac{1}{x^{-\frac{1}{x}}}$.
28. $a^{-\frac{1}{3}}\times 2a^{-\frac{1}{2}}$.
29. $x^{-\frac{2}{3}}\div 2a^{-\frac{1}{2}}$.
30. $7a^{-\frac{1}{2}}\times 3a^{-1}$.
31. $\dfrac{2a^{-2}}{a^{-\frac{3}{2}}}$.
32. $\dfrac{a^{-\frac{1}{2}}}{3a}$.
33. $\dfrac{4x^{-1}}{x^{-\frac{1}{3}}}$.
34. $\dfrac{\sqrt[3]{x^{-a}}}{\sqrt[3]{x^2}}$.
35. $\sqrt[3]{a^2}\times\sqrt[2]{a^3}$.
36. $\sqrt[5]{a^{-x}}\div\sqrt[5]{a^{-2x}}$.
37. $\sqrt[2a]{x}\times\sqrt[a]{x^2}$.
38. $\sqrt[a]{x}\div\sqrt[2a]{x^3}$.
39. $\sqrt[3x]{a^3}\div\sqrt[x]{a^2}$.
40. $\sqrt[4]{a^n}\times\sqrt[3]{a^n}\div\sqrt[12]{a^{5n}}$.

Find the value of

41. $16^{\frac{3}{4}}$. 43. $125^{\frac{2}{3}}$. 45. $36^{-\frac{3}{2}}$. 47. $243^{\frac{2}{5}}$. 49. $(\frac{81}{16})^{\frac{3}{4}}$.

42. $4^{-\frac{5}{2}}$. 44. $8^{-\frac{2}{3}}$. 46. $\frac{1}{25^{-2}}$. 48. $(\frac{8}{27})^{-\frac{1}{3}}$. 50. $(\frac{32}{243})^{-\frac{7}{5}}$.

221. To prove that $(a^m)^n = a^{mn}$ is true for all values of *m* and *n*.

Case I. Let n be a *positive integer.*

Now, *whatever be the value of* m

$$(a^m)^n = a^m \cdot a^m \cdot a^m \cdots \text{ to } n \text{ factors}$$
$$= a^{m+m+m+\cdots \text{ to } n \text{ terms}} = a^{mn}.$$

Case II. Let m be unrestricted as before, and let n be a *positive fraction.* Replacing n by $\frac{p}{q}$, where p and q are *positive integers*, we have $(a^m)^n = (a^m)^{\frac{p}{q}}$.

Now the qth power of $(a^m)^{\frac{p}{q}} = \{(a^m)^{\frac{p}{q}}\}^q = (a^m)^{\frac{p}{q} \cdot q}$, [Case I.]

$$= (a^m)^p = a^{mp}. \qquad \text{[Case I.]}$$

Hence by taking the qth root of these equals,

$$(a^m)^{\frac{p}{q}} = \sqrt[q]{a^{mp}} = a^{\frac{mp}{q}}. \qquad \text{[Art. 215.]}$$

Case III. Let m be unrestricted as before, and let n be *any negative quantity.* Replacing n by $-r$, where r is *positive*, we have

$$(a^m)^n = (a^m)^{-r} = \frac{1}{(a^m)^r}, \qquad \text{[Art. 217.]}$$
$$= \frac{1}{a^{mr}}, \qquad \text{[Case II.]}$$
$$= a^{-mr} = a^{mn}.$$

Hence Prop. III., Art. 213, $(a^m)^n = a^{mn}$ has been shown to be universally true.

EXAMPLES. (1) $(b^{\frac{2}{3}})^{\frac{6}{7}} = b^{\frac{2}{3} \times \frac{6}{7}} = b^{\frac{4}{7}}$.

(2) $\{(x^{-2})^3\}^{-4} = (x^{-6})^{-4} = x^{24}$.

(3) $(x^{\frac{1}{a-c}})^{a^2-c^2} = x^{\frac{1}{a-c} \times (a^2-c^2)} = x^{a+c}$.

222. To prove that $(ab)^n = a^n b^n$, whatever be the value of n; a and b being any quantities whatever.

Case I. Let n be a *positive integer*.

Now $(ab)^n = ab \cdot ab \cdot ab \cdots$ to n factors

$= (a \cdot a \cdot a \cdots \text{to } n \text{ factors})(b \cdot b \cdot b \cdots \text{to } n \text{ factors}) = a^n b^n.$

Case II. Let n be a *positive fraction*. Replacing n by $\frac{p}{q}$, where p and q are *positive integers*, we have $(ab)^n = (ab)^{\frac{p}{q}}$.

Now the qth power of $(ab)^{\frac{p}{q}} = \{(ab)^{\frac{p}{q}}\}^q = (ab)^p$, [Art. 221.]

$$= a^p b^p = \left(a^{\frac{p}{q}} b^{\frac{p}{q}}\right)^q. \qquad \text{[Case I.]}$$

Taking the qth root, $(ab)^{\frac{p}{q}} = a^{\frac{p}{q}} b^{\frac{p}{q}}$.

Case III. Let n have *any negative value*. Replacing n by $-r$, where r is positive,

$$(ab)^n = (ab)^{-r} = \frac{1}{(ab)^r} = \frac{1}{a^r b^r} = a^{-r} b^{-r} = a^n b^n.$$

Hence the proposition is proved universally.

This result may be expressed in a verbal form by saying that *the index of a product may be distributed over its factors*.

NOTE. An index is not distributive over the **terms** of an expression. Thus $(a^{\frac{1}{2}} + b^{\frac{1}{2}})^2$ is not equal to $a + b$. Again $(a^2 + b^2)^{\frac{1}{2}}$ is equal to $\sqrt{a^2 + b^2}$, and cannot be further simplified.

EXAMPLES. (1) $(yz)^{a-c}(zx)^c(xy)^{-c} = y^{a-c} z^{a-c} z^c x^c x^{-c} y^{-c} = y^{a-2c} z^a.$

(2) $\{(a-b)^k\}^{-l} \times \{(a+b)^{-k}\}^l = (a-b)^{-kl} \times (a+b)^{-kl}$
$= \{(a-b)(a+b)\}^{-kl} = (a^2 - b^2)^{-kl}.$

223. Since in the proof of Art. 222 the quantities a and b are *wholly unrestricted*, they may themselves involve indices.

EXAMPLES. (1) $(x^{\frac{1}{2}} y^{-\frac{1}{2}})^{\frac{4}{3}} \div (x^2 y^{-1})^{-\frac{1}{3}} = x^{\frac{2}{3}} y^{-\frac{2}{3}} \div x^{-\frac{2}{3}} y^{\frac{1}{3}} = x^{\frac{4}{3}} y^{-1}.$

(2) $\left(\dfrac{a^{\frac{2}{3}}\sqrt{b^{-1}}}{b\sqrt[3]{a^{-2}}} \div \sqrt{\dfrac{a\sqrt{b^{-4}}}{b\sqrt{a^{-2}}}}\right)^6 = \left(\dfrac{a^{\frac{2}{3}} b^{-\frac{1}{2}}}{ba^{-\frac{2}{3}}} \div \sqrt{\dfrac{ab^{-2}}{ba^{-1}}}\right)^6$

$= (a^{\frac{4}{3}} b^{-\frac{3}{2}} \div \sqrt{a^2 b^{-3}})^6 = (a^{\frac{4}{3}} b^{-\frac{3}{2}} \div ab^{-\frac{3}{2}})^6 = (a^{\frac{1}{3}})^6 = a^2.$

EXAMPLES XXII. b.

Simplify and express with positive indices:

1. $(\sqrt{a^2b^3})^6$.

2. $(\sqrt[9]{x^{-4}y^3})^{-3}$.

3. $(x^a y^{-b})^3 \times (x^3y^2)^{-a}$.

4. $\left(\dfrac{16\,x^2}{y^{-2}}\right)^{-\frac{1}{4}}$.

5. $\left(\dfrac{27\,x^3}{8\,a^{-3}}\right)^{-\frac{2}{3}}$.

6. $\left(\dfrac{a^{-\frac{1}{2}}}{4\,c^2}\right)^{-2}$.

7. $\left\{\sqrt[4]{(x^{-\frac{2}{3}}y^{\frac{1}{2}})^3}\right\}^{-\frac{2}{3}}$.

8. $\sqrt[4]{x\sqrt[3]{x^{-1}}}$.

9. $(4\,a^{-2} \div 9\,x^2)^{-\frac{1}{2}}$.

10. $(x \div \sqrt[n]{x})^n$.

11. $(x \times \sqrt[n]{x^{-\frac{1}{n}}})^{\frac{n^2}{1-n}}$.

12. $(\sqrt[b]{x^b} \div \sqrt[a]{x})^{\frac{1}{1-a}}$.

13. $\sqrt{a^{-2}b} \times \sqrt[3]{ab^{-3}}$.

14. $\sqrt[3]{ab^{-1}c^{-2}} \times (a^{-1}b^{-2}c^{-4})^{-\frac{1}{6}}$.

15. $\sqrt[6]{a^{4b}x^6} \times (a^{\frac{2}{3}}x^{-1})^{-b}$.

16. $\sqrt[3]{x^{-1}\sqrt{y^3}} \div \sqrt{y\sqrt[3]{x}}$.

17. $(a^{-\frac{1}{2}}\sqrt[3]{x})^{-3} \times \sqrt{x^{-2}\sqrt{a^{-6}}}$.

18. $\sqrt[n]{a^{n+k}b^{2n-k}} \div (a^{\frac{1}{n}}b^{-\frac{1}{n}})^k$.

19. $\sqrt[3]{(a+b)^5} \times (a+b)^{-\frac{2}{3}}$.

20. $\{(x-y)^{-3}\}^n \div \{(x+y)^n\}^3$.

21. $\left(\dfrac{a^{-2}b}{a^3b^{-4}}\right)^{-3} \div \left(\dfrac{ab^{-1}}{a^{-3}b^2}\right)^5$.

22. $\left\{\dfrac{\sqrt[3]{a}}{\sqrt[4]{b^{-1}}} \cdot \left(\dfrac{b^{\frac{1}{4}}}{a^{\frac{1}{3}}}\right)^2 \div \dfrac{a^{-\frac{1}{3}}}{b^{-\frac{1}{2}}}\right\}^6$.

23. $\left(a^{-\frac{1}{2}}x^{\frac{1}{3}}\sqrt{ax^{-\frac{1}{3}}\sqrt[4]{x^{\frac{4}{3}}}}\right)^{\frac{1}{3}}$.

24. $\sqrt[4]{(a+b)^6} \times (a^2-b^2)^{-\frac{1}{2}}$.

25. $\left(\dfrac{a^{-3}}{b^{-\frac{2}{3}}c}\right)^{-\frac{3}{2}} \div \left(\dfrac{\sqrt{a^{-\frac{1}{2}}} \cdot \sqrt[6]{b^3}}{a^2c^{-1}}\right)^{-2}$.

26. $\left(\dfrac{a^{-\frac{2}{3}}x^{\frac{1}{2}}}{x^{-1}a}\right)^2 \div \sqrt[3]{\dfrac{a^{-1}}{x^{-3}}}$.

27. $\left(\sqrt[5]{\dfrac{a^{\frac{1}{2}}x^{-2}}{x^{\frac{1}{2}}a^{-2}}} \times \sqrt[3]{\dfrac{a\sqrt{x}}{x^{-1}\sqrt{a}}}\right)^{-4}$.

28. $\dfrac{\sqrt[3]{(a^3b^3+a^6)}}{\sqrt[3]{(b^6-a^3b^3)^{-1}}}$.

29. $(a^{n^2-1})^{\frac{n}{n+1}} + \dfrac{\sqrt[n]{a^{2n}}}{a}$.

30. $(x^{\frac{n}{n+1}})^{n^2-1} + \dfrac{\sqrt{x^{2n}}}{x}$.

31. $\left\{\dfrac{a^{p-q}}{\sqrt[q]{a^{q^2-pq}}} \times a^{2(p-q)}\right\}^n$.

32. $(x^{\frac{a}{b}}y^{-1})^b \div \left(\dfrac{x^{a^2-b^2}}{y^{ab+b^2}}\right)^{\frac{1}{a+b}}$.

33. $\left(\dfrac{x^{-2}y^3}{x^3y^{-2}}\right)^{-\frac{1}{5}} \times \left(\dfrac{y^3x^{-3}}{x^3y^{-3}}\right)^{-1}$.

34. $\left(\dfrac{y^{-3}}{x^{\frac{2}{7}}z^{-1}}\right)^{-\frac{3}{2}} \times \left(\dfrac{y^{\frac{14}{3}}x^{-1}}{z^{-\frac{21}{4}}}\right)^{\frac{2}{7}}$.

35. $\dfrac{2^n \times (2^{n-1})^n}{2^{n+1} \times 2^{n-1}} \times \dfrac{1}{4^{-n}}$.

36. $\dfrac{2^{n+1}}{(2^n)^{n-1}} \div \dfrac{4^{n+1}}{(2^{n-1})^{n+1}}$.

224. Since the index-laws are universally true, all the ordinary operations of multiplication, division, involution, and evolution are applicable to expressions which contain fractional and negative indices.

225. In Art. 200, we pointed out that the descending powers of x are

$$\cdots x^3,\ x^2,\ x,\ 1,\ \frac{1}{x},\ \frac{1}{x^2},\ \frac{1}{x^3},\ \cdots.$$

A reason is seen if we write these terms in the form

$$\cdots x^3, x^2, x^1,\ x^0, x^{-1}, x^{-2}, x^{-3}, \cdots.$$

Ex. 1. Multiply $3x^{-\frac{1}{3}} + x + 2x^{\frac{2}{3}}$ by $x^{\frac{1}{3}} - 2$.

Arrange in descending powers of x.

$$\begin{array}{l}
x \ + 2x^{\frac{2}{3}} + 3x^{-\frac{1}{3}} \\
\underline{x^{\frac{1}{3}} - 2\qquad\qquad} \\
x^{\frac{4}{3}} + 2x \ + 3 \\
\underline{\qquad - 2x \ - 4x^{\frac{2}{3}} - 6x^{-\frac{1}{3}}} \\
x^{\frac{4}{3}} - 4x^{\frac{2}{3}} + 3 \qquad - 6x^{-\frac{1}{3}}
\end{array}$$

Ex. 2. Divide $16a^{-3} - 6a^{-2} + 5a^{-1} + 6$ by $1 + 2a^{-1}$.

$$\begin{array}{l}
2a^{-1} + 1)\,16a^{-3} - \ 6a^{-2} + \ 5a^{-1} + 6\,(8a^{-2} - 7a^{-1} + 6 \\
\qquad\qquad \underline{16a^{-3} + \ 8a^{-2}\qquad} \\
\qquad\qquad\qquad - 14a^{-2} + \ 5a^{-1} \\
\qquad\qquad\qquad \underline{- 14a^{-2} - \ 7a^{-1}\qquad} \\
\qquad\qquad\qquad\qquad\qquad 12a^{-1} + 6 \\
\qquad\qquad\qquad\qquad\qquad 12a^{-1} + 6
\end{array}$$

Ex. 3. Find the square root of

$$\frac{4x^2}{y} + \frac{\sqrt{x^3}}{y^{-\frac{1}{2}}} - 2x + \frac{y}{4} + x^3 - 4\sqrt{(x^5y^{-1})}.$$

Use fractional indices, and arrange in descending powers of x.

$$\begin{array}{l|l}
 & x^3 - 4x^{\frac{5}{2}}y^{-\frac{1}{2}} + 4x^2y^{-1} + x^{\frac{3}{2}}y^{\frac{1}{2}} - 2x + \frac{y}{4}\left(x^{\frac{3}{2}} - 2xy^{-\frac{1}{2}} + \frac{y^{\frac{1}{2}}}{2}\right. \\
 & \underline{x^3\qquad\qquad\qquad\qquad} \\
2x^{\frac{3}{2}} - 2xy^{-\frac{1}{2}} & - 4x^{\frac{5}{2}}y^{-\frac{1}{2}} + 4x^2y^{-1} \\
 & \underline{- 4x^{\frac{5}{2}}y^{-\frac{1}{2}} + 4x^2y^{-1}} \\
2x^{\frac{3}{2}} - 4xy^{-\frac{1}{2}} + \frac{y^{\frac{1}{2}}}{2} & \qquad\qquad\qquad x^{\frac{3}{2}}y^{\frac{1}{2}} - 2x + \frac{y}{4} \\
 & \qquad\qquad\qquad \underline{x^{\frac{3}{2}}y^{\frac{1}{2}} - 2x + \frac{y}{4}}
\end{array}$$

NOTE. In this example it should be observed that the introduction of negative indices enables us to avoid the use of algebraic fractions.

EXAMPLES XXII. c.

1. Multiply $3x^{\frac{1}{3}} - 5 + 8x^{-\frac{1}{3}}$ by $4x^{\frac{1}{3}} + 3x^{-\frac{1}{3}}$.
2. Multiply $3a^{\frac{3}{5}} - 4a^{\frac{1}{5}} - a^{-\frac{1}{5}}$ by $3a^{\frac{1}{5}} + a^{-\frac{1}{5}} - 6a^{-\frac{3}{5}}$.
3. Find the product of $c^x + 2c^{-x} - 7$ and $5 - 3c^{-x} + 2c^x$.
4. Find the product of $5 + 2x^{2a} + 3x^{-2a}$ and $4x^a - 3x^{-a}$.
5. Divide $21x + x^{\frac{2}{3}} + x^{\frac{1}{3}} + 1$ by $3x^{\frac{1}{3}} + 1$.
6. Divide $15a - 3a^{\frac{1}{3}} - 2a^{-\frac{1}{3}} + 8a^{-1}$ by $5a^{\frac{2}{3}} + 4$.
7. Divide $16a^{-3} + 6a^{-2} + 5a^{-1} - 6$ by $2a^{-1} - 1$.
8. Divide $5b^{\frac{2}{3}} - 6b^{\frac{1}{3}} - 4b^{-\frac{2}{3}} - 4b^{-\frac{1}{3}} - 5$ by $b^{\frac{1}{6}} - 2b^{-\frac{1}{6}}$.
9. Divide $21a^{3x} + 20 - 27a^x - 26a^{2x}$ by $3a^x - 5$.
10. Divide $8c^{-n} - 8c^n + 5c^{3n} - 3c^{-3n}$ by $5c^n - 3c^{-n}$.

Find the square root of

11. $9x - 12x^{\frac{1}{2}} + 10 - 4x^{-\frac{1}{2}} + x^{-1}$.
12. $25a^{\frac{4}{3}} + 16 - 30a - 24a^{\frac{1}{3}} + 49a^{\frac{2}{3}}$.
13. $4x^n + 9x^{-n} + 28 - 24x^{-\frac{n}{2}} - 16x^{\frac{n}{2}}$.
14. $12a^x + 4 - 6a^{3x} + a^{4x} + 5a^{2x}$.
15. Multiply $a^{\frac{3}{2}} - 8a^{-\frac{3}{2}} + 4a^{-\frac{1}{2}} - 2a^{\frac{1}{2}}$ by $4a^{-\frac{3}{2}} + a^{\frac{1}{2}} + 4a^{-\frac{1}{2}}$.
16. Multiply $1 - 2\sqrt[3]{x} - 2x^{\frac{1}{2}}$ by $1 - \sqrt[6]{x}$.
17. Multiply $2\sqrt[3]{a^5} - a^{\frac{1}{3}} - \dfrac{3}{a}$ by $2a - 3\sqrt[3]{\dfrac{1}{a}} - a^{-\frac{5}{3}}$.
18. Divide $\sqrt[3]{x^2} + 2x^{\frac{1}{3}} - 16x^{-\frac{2}{3}} - \dfrac{32}{x}$ by $x^{\frac{1}{6}} + 4x^{-\frac{1}{6}} + \dfrac{4}{\sqrt{x}}$.
19. Divide $1 - \sqrt{a} - \dfrac{2}{a^{-1}} + 2a^2$ by $1 - a^{\frac{1}{2}}$.
20. Divide $4\sqrt[3]{x^2} - 8x^{\frac{1}{3}} - 5 + \dfrac{10}{\sqrt[3]{x}} + 3x^{-\frac{2}{3}}$ by $2x^{\frac{5}{12}} - \sqrt[12]{x} - \dfrac{3}{\sqrt[4]{x}}$.

Find the square root of

21. $9x^{-4} - 18x^{-3}\sqrt{y} + \dfrac{15y}{x^2} - 6\sqrt{\left(\dfrac{y^3}{x^2}\right)} + y^2$.
22. $4\sqrt{x^3} - 12\sqrt[4]{(x^3y)} + 25\sqrt{y} - 24\sqrt[4]{\left(\dfrac{y^3}{x^3}\right)} + 16x^{-\frac{3}{2}}y$
23. $81\left(\dfrac{\sqrt[3]{x^4}}{y^2} + 1\right) + 36\dfrac{x^{\frac{1}{3}}}{\sqrt{y}}(x^{\frac{2}{3}}y^{-1} - 1) - 158\dfrac{\sqrt[3]{x^2}}{y}$.
24. $\dfrac{x^{-2}}{16} + 1 + \dfrac{9}{\sqrt[3]{y^{-2}}} + \dfrac{1 - 3\sqrt[3]{y}}{2x} - 6\sqrt[3]{y}$.

226. The following examples will illustrate the formulæ of earlier chapters when applied to expressions involving fractional and negative indices.

Ex. 1. $(a^{\frac{h}{k}} - b^{\frac{p}{q}})(a^{-\frac{h}{k}} + b^{-\frac{p}{q}}) = a^{\frac{h}{k}-\frac{h}{k}} - a^{-\frac{h}{k}}b^{\frac{p}{q}} + a^{\frac{h}{k}}b^{-\frac{p}{q}} - b^{\frac{p}{q}-\frac{p}{q}}$

$$= 1 - a^{-\frac{h}{k}}b^{\frac{p}{q}} + a^{\frac{h}{k}}b^{-\frac{p}{q}} - 1$$

$$= a^{\frac{h}{k}}b^{-\frac{p}{q}} - a^{-\frac{h}{k}}b^{\frac{p}{q}}.$$

Ex. 2. Multiply $2x^{2p} - x^p + 3$ by $2x^{2p} + x^p - 3$.

The product $= \{2x^{2p} - (x^p - 3)\}\{2x^{2p} + (x^p - 3)\}$

$$= (2x^{2p})^2 - (x^p - 3)^2 = 4x^{4p} - x^{2p} + 6x^p - 9.$$

Ex. 3. The square of $3x^{\frac{1}{2}} - 2 - x^{-\frac{1}{2}}$

$$= 9x + 4 + x^{-1} - 2\cdot 3x^{\frac{1}{2}}\cdot 2 - 2\cdot 3x^{\frac{1}{2}}\cdot x^{-\frac{1}{2}} + 2\cdot 2\cdot x^{-\frac{1}{2}}$$

$$= 9x + 4 + x^{-1} - 12x^{\frac{1}{2}} - 6 + 4x^{-\frac{1}{2}}$$

$$= 9x - 12x^{\frac{1}{2}} - 2 + 4x^{-\frac{1}{2}} + x^{-1}.$$

Ex. 4. Divide $a^{\frac{3n}{2}} + a^{-\frac{3n}{2}}$ by $a^{\frac{n}{2}} + a^{-\frac{n}{2}}$.

The quotient $= (a^{\frac{3n}{2}} + a^{-\frac{3n}{2}}) \div (a^{\frac{n}{2}} + a^{-\frac{n}{2}})$

$$= \{(a^{\frac{n}{2}})^3 + (a^{-\frac{n}{2}})^3\} \div (a^{\frac{n}{2}} + a^{-\frac{n}{2}})$$

$$= (a^{\frac{n}{2}})^2 - a^{\frac{n}{2}}\cdot a^{-\frac{n}{2}} + (a^{-\frac{n}{2}})^2 = a^n - 1 + a^{-n}.$$

EXAMPLES XXII. d.

Write the value of

1. $(x^{\frac{1}{2}} - 7)(x^{\frac{1}{2}} + 3)$.
2. $(4x - 5x^{-1})(4x + 3x^{-1})$.
3. $(7x - 9y^{-1})(7x + 9y^{-1})$.
4. $(x^m - y^n)(x^{-m} + y^{-n})$.
5. $(a^x - 2a^{-x})^2$.
6. $(a^x + a^{\frac{1}{x}})^2$.
7. $(x^{\frac{a}{2}} - \frac{1}{2}x^{-a})^2$.
8. $(5x^ay^b - 3x^{-a}y^{-b})(4x^ay^b + 5x^{-a}y^{-b})$.
9. $(\frac{1}{3}a^{\frac{1}{3}} - a^{-\frac{1}{3}})^2$.
10. $(3x^ay^{-b} + 5x^{-a}y^b)(3x^ay^b - 5x^{-a}y^{-b})$.

11. $(a^x - \frac{1}{2} - a^{-x})^2$.

12. $(x^{\frac{1}{a}} - x^{-\frac{1}{a}} + x)^2$.

13. $\{(a+b)^{\frac{1}{2}} + (a-b)^{\frac{1}{2}}\}^2$.

14. $\{(a+b)^{\frac{1}{2}} - (a-b)^{-\frac{1}{2}}\}^2$.

Write the quotient of

15. $x - 9a$ by $x^{\frac{1}{2}} + 3a^{\frac{1}{2}}$.

16. $x^{\frac{3}{2}} - 27$ by $x^{\frac{1}{2}} - 3$.

17. $a^{2x} - 16$ by $a^x - 4$.

18. $x^{3a} + 8$ by $x^a + 2$.

19. $c^{2x} - c^{-x}$ by $c^x - c^{-\frac{x}{2}}$.

20. $1 - 8a^{-3}$ by $1 - 2a^{-1}$.

21. $a^{4x} - x^6$ by $a^{2x} + x^3$.

22. $x^{-4} - 1$ by $x^{-1} + 1$.

23. $x^{\frac{5}{3}} - 1$ by $x^{\frac{1}{3}} - 1$.

24. $x^{5n} + 32$ by $x^n + 2$.

Find the value of

25. $(x + x^{\frac{1}{2}} - 4)(x + x^{\frac{1}{2}} + 4)$.

26. $(2x^{\frac{1}{3}} + 4 + 3x^{-\frac{1}{3}})(2x^{\frac{1}{3}} + 4 - 3x^{-\frac{1}{3}})$.

27. $(2 - x^{\frac{1}{3}} + x)(2 + x^{\frac{1}{3}} + x)$.

28. $(a^x + 7 + 3a^{-x})(a^x - 7 - 3a^{-x})$.

29. $\dfrac{a^{\frac{4}{3}} - 8a^{\frac{1}{3}}b}{a^{\frac{2}{3}} + 2\sqrt[3]{ab} + 4b^{\frac{2}{3}}}$.

30. $\dfrac{x - 7x^{\frac{1}{2}}}{x - 5\sqrt{x} - 14} \div \left(1 + \dfrac{2}{\sqrt{x}}\right)^{-1}$.

31. $\dfrac{x^{\frac{2}{3}} - 4\sqrt[3]{x^{-2}}}{\sqrt[3]{x^2} + 4 + 4x^{-\frac{2}{3}}}$.

32. $\dfrac{a^{\frac{3}{2}} + ab}{ab - b^3} - \dfrac{\sqrt{a}}{\sqrt{a} - b}$.

CHAPTER XXIII.

SURDS (RADICALS).

227. A **surd** is an indicated root which cannot be exactly obtained.

Thus $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[5]{a^3}$, $\sqrt{a^2+b^2}$ are surds.

By reference to the preceding chapter it will be seen that these are only cases of fractional indices; for the above quantities might be written

$$2^{\frac{1}{2}},\ 5^{\frac{1}{3}},\ a^{\frac{3}{5}},\ (a^2+b^2)^{\frac{1}{2}}.$$

Since surds may always be expressed as quantities with fractional indices they are subject to the same laws of combination as other algebraic symbols.

228. A surd is sometimes called an **irrational quantity**; and quantities which are not surds are, for the sake of distinction, termed **rational quantities.**

229. Surds are sometimes spoken of as **radicals.** This term is also applied to quantities such as $\sqrt{a^2}$, $\sqrt{9}$, $\sqrt[3]{27}$, etc., which are, however, *rational quantities in surd form.*

230. The **order** of a surd is indicated by the root symbol, or surd index. Thus $\sqrt[3]{x}$, $\sqrt[n]{a}$ are respectively surds of the third and nth orders.

The surds of the most frequent occurrence are those of the second order; they are sometimes called **quadratic surds.** Thus $\sqrt{3}$, $\sqrt{a}$, $\sqrt{x+y}$ are quadratic surds.

231. A **mixed surd** is one containing a factor whose root can be extracted.

This factor can evidently be removed and its root placed before the radical as a **coefficient**. It is called the **rational factor**, and the factor whose root cannot be extracted is called the **irrational factor**.

232. When the coefficient of the surd is unity, it is said to be **entire**.

233. When the irrational factor is integral, and all rational factors have been removed, the surd is in its **simplest form**.

234. When surds of the *same order* contain the same *irrational factor*, they are said to be **similar** or **like**.

Thus $5\sqrt{3}$, $2\sqrt{3}$, $\frac{1}{5}\sqrt{3}$ are *like* surds.

But $3\sqrt{2}$ and $2\sqrt{3}$ are *unlike* surds.

235. In the case of numerical surds such as $\sqrt{2}$, $\sqrt[3]{5}$, $\cdots$, although the *exact* value can never be found, it can be determined to any degree of accuracy by carrying the process of evolution far enough.

Thus $$\sqrt{5} = 2{\cdot}236068\cdots;$$

that is $\sqrt{5}$ lies between 2.23606 and 2.23607; and therefore the error in using either of these quantities instead of $\sqrt{5}$ is less than .00001. By taking the root to a greater number of decimal places we can approximate still nearer to the true value.

It thus appears that it will never be *absolutely necessary* to introduce surds into numerical work, which can always be carried on to a certain degree of accuracy; but we shall in the present chapter prove laws for combination of surd quantities which will enable us to work with symbols such as $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[4]{a}$, $\cdots$ with absolute accuracy so long as the symbols are kept in their surd form. Moreover it will be found that even where approximate numerical results are required, the work is considerably simplified and shortened by operating with surd symbols, and afterwards substituting numerical values, if necessary.

REDUCTION OF SURDS.

236. Transformation of Surds of Any Order into Surds of a Different Order having the Same Value. A surd of any order may be transformed into a surd of a different order having the same value. Such surds are said to be **equivalent.**

EXAMPLES. (1) $\sqrt[3]{2} = 2^{\frac{1}{3}} = 2^{\frac{4}{12}} = \sqrt[12]{2^4}$.

(2) $\sqrt[p]{a} = a^{\frac{1}{p}} = a^{\frac{q}{pq}} = \sqrt[pq]{a^q}$.

237. Surds of different orders may therefore be transformed into surds of the same order. This order may be *any* common multiple of each of the given orders, but it is usually most convenient to choose the *least* common multiple.

Ex. Express $\sqrt[4]{a^3}$, $\sqrt[3]{b^2}$, $\sqrt[6]{a^5}$ as surds of the same lowest order.

The least common multiple of 4, 3, 6 is 12; and expressing the given surds as surds of the twelfth order they become $\sqrt[12]{a^9}$, $\sqrt[12]{b^8}$, $\sqrt[12]{a^{10}}$.

238. Surds of different orders may be arranged according to magnitude by transforming them into surds of the same order.

Ex. Arrange $\sqrt{3}$, $\sqrt[3]{6}$, $\sqrt[4]{10}$ according to magnitude.

The least common multiple of 2, 3, 4 is 12; and, expressing the given surds as surds of the twelfth order, we have

$$\sqrt{3} = \sqrt[12]{3^6} = \sqrt[12]{729},$$
$$\sqrt[3]{6} = \sqrt[12]{6^4} = \sqrt[12]{1296},$$
$$\sqrt[4]{10} = \sqrt[12]{10^3} = \sqrt[12]{1000}.$$

Hence arranged in ascending order of magnitude the surds are

$$\sqrt{3},\ \sqrt[4]{10},\ \sqrt[3]{6}.$$

EXAMPLES XXIII. a.

Express as surds of the twelfth order with positive indices:

1. $x^{\frac{1}{3}}$. **2.** $a^{-1} \div a^{-\frac{1}{2}}$. **3.** $\sqrt[4]{ax^3} \times \sqrt[6]{a^{-1}x^{-2}}$.

4. $\dfrac{1}{a^{-\frac{3}{4}}}$. **5.** $\dfrac{1}{\sqrt[8]{a^{-14}}}$. **6.** $\sqrt[6]{\dfrac{1}{a^{-2}}}$.

Express as surds of the nth order with positive indices:

7. $\sqrt[3]{x^2}$. 9. $a^{\frac{1}{2}}$. 11. $\sqrt[3]{\frac{1}{x^n y^n}}$. 13. $\frac{x^{-\frac{1}{2}}}{y^2}$.

8. x^a. 10. $\sqrt{a^{-\frac{1}{n}}}$. 12. $\frac{1}{a^{-1}}$. 14. $\frac{a^{\frac{1}{2}}}{x^{-n}}$.

Express as surds of the same lowest order:

15. $\sqrt{a}$, $\sqrt[9]{a^5}$. 18. $\sqrt[16]{x^4}$, $\sqrt[12]{x^{10}}$. 21. $\sqrt{5}$, $\sqrt[3]{11}$, $\sqrt[6]{13}$.

16. $\sqrt[5]{a^3}$, $\sqrt{a}$. 19. $\sqrt[21]{a^3b^4}$, $\sqrt[7]{ab}$. 22. $\sqrt[4]{8}$, $\sqrt{3}$, $\sqrt[8]{6}$.

17. $\sqrt[8]{x^3}$, $\sqrt[9]{x^6}$, $\sqrt[20]{x^5}$. 20. $\sqrt{ax^2}$, $\sqrt[39]{a^9x^6}$. 23. $\sqrt[3]{2}$, $\sqrt[9]{8}$, $\sqrt[6]{4}$.

239. Reduction of a Surd to its Simplest Form. The root of any expression is equal to the product of the roots of the separate factors of the expression.

For $$\sqrt[n]{ab} = (ab)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}, \qquad \text{[Art. 222.]}$$
$$= \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

Similarly, $$\sqrt[n]{abc} = \sqrt[n]{a} \cdot \sqrt[n]{b} \cdot \sqrt[n]{c};$$

and so for any number of factors.

Examples. (1) $\sqrt[4]{15} = \sqrt[4]{3} \cdot \sqrt[4]{5}$.

(2) $\sqrt[3]{a^6b} = \sqrt[3]{a^6} \cdot \sqrt[3]{b} = a^2\sqrt[3]{b}$.

(3) $\sqrt{50} = \sqrt{25} \cdot \sqrt{2} = 5\sqrt{2}$.

Hence it appears that a surd may sometimes be expressed as the product of a rational quantity and a surd; when the surd factor is integral and as small as possible, the surd is in its *simplest form* [Art. 233].

Thus the simplest form of $\sqrt{128}$ is $8\sqrt{2}$.

Conversely, the coefficient of a surd may be brought under the radical sign by first raising it to the power whose root the surd expresses, and then placing the product of this power and the surd factor under the radical sign.

Examples. (1) $7\sqrt{5} = \sqrt{49} \cdot \sqrt{5} = \sqrt{245}$.

(2) $a\sqrt[8]{b} = \sqrt[8]{a^8} \cdot \sqrt[8]{b} = \sqrt[8]{a^8b}$.

In this form a surd is said to be an *entire surd* [Art. 232].

By the same method any rational quantity may be expressed in the form of a surd. Thus 2 may be written as $\sqrt{4}$, and 3 as $\sqrt[3]{27}$.

240. When the surd has the form of a fraction, we multiply both numerator and denominator by such a quantity as will make the denominator a perfect power of the same degree as the surd, and then take out the rational factor as a coefficient.

EXAMPLES. (1) $\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \sqrt{2 \times \frac{1}{4}} = \frac{1}{2}\sqrt{2}.$

(2) $\sqrt{\dfrac{mx}{ab}} = \sqrt{\dfrac{abmx}{a^2b^2}} = \dfrac{1}{ab}\sqrt{abmx}.$

EXAMPLES XXIII. b.

Express in the simplest form:

1. $\sqrt{288}$.
2. $\sqrt{147}$.
3. $\sqrt[3]{256}$.
4. $\sqrt[3]{432}$.
5. $3\sqrt{150}$.
6. $2\sqrt{720}$.
7. $5\sqrt{245}$.
8. $\sqrt[3]{1029}$.
9. $\sqrt[4]{3125}$.
10. $\sqrt[3]{-2187}$.
11. $\sqrt{36a^3}$.
12. $\sqrt{27a^3b^5}$.
13. $\sqrt{\frac{2}{7}}$.
14. $\sqrt[3]{\frac{3}{4}}$.
15. $\sqrt{\dfrac{2a}{b}}$.
16. $\sqrt[3]{\frac{14}{27}}$.
17. $\sqrt[3]{-108x^4y^3}$.
18. $\sqrt[n]{x^{3n}y^{2n+5}}$.
19. $\sqrt[p]{x^{a+p}y^{2p}}$.
20. $\sqrt{a^3 + 2a^2b + ab^2}$.
21. $\sqrt[3]{8x^4y - 24x^3y^2 + 24x^2y^3 - 8xy^4}$.

Express (1) as entire surds, (2) in simplest form:

22. $11\sqrt{2}$.
23. $14\sqrt{5}$.
24. $6\sqrt[3]{4}$.
25. $5\sqrt[3]{6}$.
26. $\frac{4}{11}\sqrt{\frac{77}{8}}$.
27. $\dfrac{3ab}{2c}\sqrt{\dfrac{20c^2}{9a^2b}}$.
28. $\dfrac{3x}{y}\sqrt{\dfrac{a^2y^3}{x^2}}$.
29. $\dfrac{a}{x^2}\sqrt{\dfrac{3x^3}{a}}$.
30. $\dfrac{2a}{3x}\sqrt[3]{\dfrac{27x^4}{a^2}}$.
31. $\dfrac{2a}{b}\sqrt[4]{\dfrac{b^4}{8a^3}}$.
32. $a\sqrt[n]{\dfrac{b^2}{a^{n-2}}}$.
33. $\dfrac{a}{b}\sqrt[p]{\dfrac{b^{p+1}}{a^{p-1}}}$.
34. $\dfrac{y}{x^n}\sqrt{\dfrac{x^{2n+1}}{y^3}}$.
35. $(x+y)\sqrt{\dfrac{x-y}{x+y}}$.
36. $\dfrac{ax}{a-x}\sqrt{\dfrac{a^2-x^2}{a^2x^2}}$.

ADDITION AND SUBTRACTION OF SURDS.

241. To add and subtract like surds: *Reduce them to their simplest form, and prefix to their common irrational part the sum of the coefficients.*

Ex. 1. The sum of $3\sqrt{20}$, $4\sqrt{5}$, $\dfrac{\sqrt{5}}{5}$

$$=6\sqrt{5}+4\sqrt{5}+\frac{1}{5}\sqrt{5}=\frac{51}{5}\sqrt{5}.$$

Ex. 2. The sum of $x\sqrt[3]{8x^3a}+y\sqrt[3]{-y^3a}-z\sqrt[3]{z^3a}$

$$=x\cdot 2x\sqrt[3]{a}+y(-y)\sqrt[3]{a}-z\cdot z\sqrt[3]{a}$$
$$=(2x^2-y^2-z^2)\sqrt[3]{a}.$$

242. **Unlike surds** *cannot be collected.*

Thus the sum of $5\sqrt{2}$, $-2\sqrt{3}$, and $\sqrt{6}$ is $5\sqrt{2}-2\sqrt{3}+\sqrt{6}$, and cannot be further simplified.

EXAMPLES XXIII. c.

Find the value of

1. $3\sqrt{45}-\sqrt{20}+7\sqrt{5}$.
2. $4\sqrt{63}+5\sqrt{7}-8\sqrt{28}$.
3. $\sqrt{44}-5\sqrt{176}+2\sqrt{99}$.
4. $2\sqrt{363}-5\sqrt{243}+\sqrt{192}$.
5. $2\sqrt[3]{189}+3\sqrt[3]{875}-7\sqrt[3]{56}$.
6. $5\sqrt[3]{81}-7\sqrt[3]{192}+4\sqrt[3]{648}$.
7. $3\sqrt[4]{162}-7\sqrt[4]{32}+\sqrt[4]{1250}$.
8. $5\sqrt[3]{-54}-2\sqrt[3]{-16}+4\sqrt[3]{686}$.
9. $4\sqrt{128}+4\sqrt{75}-5\sqrt{162}$.
10. $5\sqrt{24}-2\sqrt{54}-\sqrt{6}$.
11. $\sqrt{252}-\sqrt{294}-48\sqrt{\frac{1}{6}}$.
12. $3\sqrt{147}-\frac{7}{3}\sqrt{\frac{1}{3}}-\sqrt{\frac{1}{27}}$.

MULTIPLICATION OF SURDS.

243. **To multiply two surds of the same order:** *Multiply separately the rational factors and the irrational factors.*

For $$a\sqrt[n]{x}\times b\sqrt[n]{y}=ax^{\frac{1}{n}}\times by^{\frac{1}{n}}=abx^{\frac{1}{n}}y^{\frac{1}{n}}$$
$$=ab\,(xy)^{\frac{1}{n}}=ab\sqrt[n]{xy}.$$

Examples. (1) $5\sqrt{3}\times 3\sqrt{7}=15\sqrt{21}$.

(2) $2\sqrt{x}\times 3\sqrt{x}=6x$.

(3) $\sqrt[4]{a+b}\times\sqrt[4]{a-b}=\sqrt[4]{(a+b)(a-b)}=\sqrt[4]{a^2-b^2}$.

If the surds are not in their simplest form, it will save labor to reduce them to this form before multiplication.

Ex. The product of $5\sqrt{32}$, $\sqrt{48}$, $2\sqrt{54}$
$$=5\cdot 4\sqrt{2}\times 4\sqrt{3}\times 2\cdot 3\sqrt{6}=480\cdot\sqrt{2}\cdot\sqrt{3}\cdot\sqrt{6}=480\times 6=2880.$$

244. **To multiply surds of different orders:** *Reduce them to equivalent surds of the same order, and proceed as before.*

Ex. Multiply $5\sqrt[3]{2}$ by $2\sqrt{5}$.

The product $= 5\sqrt[6]{2^2} \times 2\sqrt[6]{5^3} = 10\sqrt[6]{2^2 \times 5^3} = 10\sqrt[6]{500}$.

EXAMPLES XXIII. d.

Find the value of

1. $2\sqrt{14} \times \sqrt{21}$.
2. $3\sqrt{8} \times \sqrt{6}$.
3. $5\sqrt{a} \times 2\sqrt{3}$.
4. $2\sqrt{15} \times 3\sqrt{5}$.
5. $8\sqrt{12} \times 3\sqrt{24}$.
6. $\sqrt[3]{x+2} \times \sqrt[3]{x-2}$.
7. $\sqrt[3]{168} \times \sqrt[3]{147}$.
8. $5\sqrt[3]{128} \times 2\sqrt[3]{432}$.
9. $a\sqrt{b^3} \times b^2\sqrt{a}$.
10. $\frac{2}{3}\sqrt[4]{6} \times \sqrt[3]{3}$.
11. $\frac{3}{x}\sqrt{\frac{a^2}{x}} \times \frac{4}{3}\sqrt{\frac{x^3}{2a^4}}$.
12. $\frac{1}{2}\sqrt{\frac{a}{b}} \times \frac{1}{3}\sqrt[5]{\frac{b^2}{a^2}}$.

DIVISION OF SURDS.

245. Suppose it is required to find the numerical value of the quotient when $\sqrt{5}$ is divided by $\sqrt{7}$.

At first sight it would seem that we must find the square root of 5, which is $2.236\cdots$, and then the square root of 7, which is $2.645\cdots$, and finally divide $2.236\cdots$ by $2.645\cdots$; three troublesome operations.

But we may avoid much of this labor by multiplying both numerator and denominator by $\sqrt{7}$, so as to make the denominator a rational quantity. Thus

$$\frac{\sqrt{5}}{\sqrt{7}} = \frac{\sqrt{5}}{\sqrt{7}} \times \frac{\sqrt{7}}{\sqrt{7}} = \frac{\sqrt{5 \times 7}}{7} = \frac{\sqrt{35}}{7}.$$

Now $$\sqrt{35} = 5.916\cdots$$

$$\therefore \frac{\sqrt{5}}{\sqrt{7}} = \frac{5.916\cdots}{7} = .845\cdots$$

246. The great utility of this artifice in calculating the numerical value of surd fractions suggests its convenience in the case of *all* surd fractions, even where numerical values are not required. Thus it is usual to simplify $\frac{a\sqrt{b}}{\sqrt{c}}$ as follows:

$$\frac{a\sqrt{b}}{\sqrt{c}}=\frac{a\sqrt{b}\times\sqrt{c}}{\sqrt{c}\times\sqrt{c}}=\frac{a\sqrt{bc}}{c}.$$

The process by which surds are removed from the denominator of any fraction is known as **rationalizing the denominator.** It is effected by multiplying both numerator and denominator by any factor which renders the denominator rational. We shall return to this point in Art. 250.

247. To divide surds: *Express the result as a fraction and rationalize the denominator.*

Ex. 1. Divide $4\sqrt{75}$ by $25\sqrt{56}$.

The quotient
$$=\frac{4\sqrt{75}}{25\sqrt{56}}=\frac{4\times5\sqrt{3}}{25\times2\sqrt{14}}=\frac{2\sqrt{3}}{5\sqrt{14}}$$
$$=\frac{2\sqrt{3}\times\sqrt{14}}{5\sqrt{14}\times\sqrt{14}}=\frac{2\sqrt{42}}{5\times14}=\frac{\sqrt{42}}{35}.$$

Ex. 2.
$$\frac{\sqrt[3]{b}}{\sqrt[3]{c^2}}=\frac{\sqrt[3]{b}\times\sqrt[3]{c}}{\sqrt[3]{c^2}\times\sqrt[3]{c}}=\frac{\sqrt[3]{bc}}{\sqrt[3]{c^3}}=\frac{\sqrt[3]{bc}}{c}.$$

EXAMPLES XXIII. e.

Find the value of

1. $\sqrt{10}\div\sqrt{2}$.
2. $3\sqrt{7}\div2\sqrt{8}$.
3. $2\sqrt{120}\div\sqrt{3}$.
4. $21\sqrt{384}\div8\sqrt{98}$.
5. $5\sqrt{27}\div3\sqrt{24}$.
6. $-13\sqrt{125}\div5\sqrt{65}$.
7. $6\sqrt{14}\div2\sqrt{21}$.
8. $\frac{3\sqrt{11}}{2\sqrt{98}}\div\frac{5}{7\sqrt{22}}$.
9. $\frac{3\sqrt{48}}{5\sqrt{112}}\div\frac{6\sqrt{84}}{\sqrt{392}}$.
10. $\frac{3}{a-b}\sqrt{\frac{2x}{a-b}}\div\sqrt{\frac{18x^3}{(a-b)^5}}$.

Given $\sqrt{2}=1.41421$, $\sqrt{3}=1.73205$, $\sqrt{5}=2.23607$, $\sqrt{6}=2.44949$, $\sqrt{7}=2.64575$: find to four decimal places the numerical value of

11. $\frac{14}{\sqrt{2}}$.
12. $\frac{25}{\sqrt{5}}$.
13. $\frac{10}{\sqrt{7}}$.
14. $\frac{48}{\sqrt{6}}$.
15. $\frac{60}{\sqrt{5}}$.
16. $144\div\sqrt{6}$.
17. $\sqrt{2}\div\sqrt{3}$.
18. $\frac{1}{2\sqrt{3}}$.
19. $\frac{1}{\sqrt{500}}$.
20. $\frac{4}{\sqrt{243}}$.
21. $\frac{25}{\sqrt{252}}$.
22. $\sqrt{\frac{256}{1575}}$.
23. $\frac{3}{2\sqrt{96}}$.

COMPOUND SURDS.

248. Hitherto we have confined our attention to **simple surds,** such as $\sqrt[4]{5}$, $\sqrt[3]{a}$, $\sqrt{x+y}$. An expression involving two or more simple surds is called a **compound surd;** thus $2\sqrt{a}-3\sqrt{b}$; $\sqrt[3]{a}+\sqrt[4]{b}$ are compound surds. A binomial, which has a surd in one or both of the terms, is called a **binomial surd.**

249. Multiplication of Compound Surds. We proceed as in the multiplication of compound algebraic expressions.

Ex. 1. Multiply $2\sqrt{x}-5$ by $3\sqrt{x}$.

The product $=3\sqrt{x}(2\sqrt{x}-5)=6x-15\sqrt{x}$.

Ex. 2. Multiply $2\sqrt{5}+3\sqrt{x}$ by $\sqrt{5}-\sqrt{x}$.

$$\begin{aligned}\text{The product } &= (2\sqrt{5}+3\sqrt{x})(\sqrt{5}-\sqrt{x})\\ &= 2\sqrt{5}\cdot\sqrt{5}+3\sqrt{5}\cdot\sqrt{x}-2\sqrt{5}\cdot\sqrt{x}-3\sqrt{x}\cdot\sqrt{x}\\ &= 10-3x+\sqrt{5x}.\end{aligned}$$

Ex. 3. Find the square of $2\sqrt{x}+\sqrt{7-4x}$.

$$\begin{aligned}(2\sqrt{x}+\sqrt{7-4x})^2 &= (2\sqrt{x})^2+(\sqrt{7-4x})^2+4\sqrt{x}\cdot\sqrt{7-4x}\\ &= 4x+7-4x+4\sqrt{7x-4x^2}\\ &= 7+4\sqrt{7x-4x^2}.\end{aligned}$$

EXAMPLES XXIII. f.

Find the value of

1. $(3\sqrt{x}-5)\times 2\sqrt{x}$.
2. $(\sqrt{x}-\sqrt{a})\times 2\sqrt{x}$.
3. $(\sqrt{a}+\sqrt{b})\times\sqrt{ab}$.
4. $(\sqrt{x+y}-1)\times\sqrt{x+y}$.
5. $(2\sqrt{3}+3\sqrt{2})^2$.
6. $(\sqrt{7}+5\sqrt{3})(2\sqrt{7}-4\sqrt{3})$.
7. $(3\sqrt{5}-4\sqrt{2})(2\sqrt{5}+3\sqrt{2})$.
8. $(3\sqrt{a}-2\sqrt{x})(2\sqrt{a}+3\sqrt{x})$.
9. $(\sqrt{x}+\sqrt{x-1})\times\sqrt{x-1}$.
10. $(\sqrt{x+a}-\sqrt{x-a})\times\sqrt{x+a}$.
11. $(\sqrt{a+x}-2\sqrt{a})^2$.
12. $(2\sqrt{a}-\sqrt{1+4a})^2$.
13. $(\sqrt{a+x}-\sqrt{a-x})^2$.
14. $(\sqrt{a+x}-2)(\sqrt{a+x}-1)$.
15. $(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{2}+\sqrt{3}+\sqrt{5})$.
16. $(\sqrt{5}+3\sqrt{2}+\sqrt{7})(\sqrt{5}+3\sqrt{2}-\sqrt{7})$.

Write the square of

17. $\sqrt{2x+a}-\sqrt{2x-a}$.

18. $\sqrt{x^2-2y^2}+\sqrt{x^2+2y^2}$.

19. $\sqrt{m+n}+\sqrt{m-n}$.

20. $3\sqrt{a^2+b^2}-2\sqrt{a^2-b^2}$.

21. $3x\sqrt{2}-3\sqrt{7-2x^2}$.

22. $\sqrt{4x^2+1}-\sqrt{4x^2-1}$.

250. If we multiply together the sum and the difference of any two quadratic surds, we obtain a *rational product.* This result should be carefully noted.

EXAMPLES. (1) $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})=(\sqrt{a})^2-(\sqrt{b})^2=a-b$.

(2) $(3\sqrt{5}+4\sqrt{3})(3\sqrt{5}-4\sqrt{3})=(3\sqrt{5})^2-(4\sqrt{3})^2=45-48=-3$.

Similarly, $(4-\sqrt{a+b})(4+\sqrt{a+b})=(4)^2-(\sqrt{a+b})^2=16-a-b$.

251. DEFINITION. When two binomial quadratic surds differ only in the sign which connects their terms, they are said to be **conjugate.**

Thus $3\sqrt{7}+5\sqrt{11}$ is conjugate to $3\sqrt{7}-5\sqrt{11}$.

Similarly, $a-\sqrt{a^2-x^2}$ is conjugate to $a+\sqrt{a^2-x^2}$.

The product of two conjugate surds is rational. [Art. 250.]

Ex. $$(3\sqrt{a}+\sqrt{x-9a})(3\sqrt{a}-\sqrt{x-9a})$$
$$=(3\sqrt{a})^2-(\sqrt{x-9a})^2=9a-(x-9a)=18a-x.$$

252. Division of Compound Surds. If the divisor is a binomial quadratic surd, express the division by means of a fraction, and rationalize the denominator by multiplying numerator and denominator by the surd which is conjugate to the divisor.

Ex. 1. Divide $4+3\sqrt{2}$ by $5-3\sqrt{2}$.

The quotient $$=\frac{4+3\sqrt{2}}{5-3\sqrt{2}}=\frac{4+3\sqrt{2}}{5-3\sqrt{2}}\times\frac{5+3\sqrt{2}}{5+3\sqrt{2}}$$
$$=\frac{20+18+12\sqrt{2}+15\sqrt{2}}{25-18}=\frac{38+27\sqrt{2}}{7}.$$

Ex. 2. Rationalize the denominator of $\dfrac{b^2}{\sqrt{a^2+b^2}+a}$.

The expression $$=\frac{b^2}{\sqrt{a^2+b^2}+a}\times\frac{\sqrt{a^2+b^2}-a}{\sqrt{a^2+b^2}-a}$$
$$=\frac{b^2\{\sqrt{a^2+b^2}-a\}}{(a^2+b^2)-a^2}=\sqrt{a^2+b^2}-a.$$

Ex. 3. Divide $\dfrac{\sqrt{3}+\sqrt{2}}{2-\sqrt{3}}$ by $\dfrac{7+4\sqrt{3}}{\sqrt{3}-\sqrt{2}}$.

$$\text{The quotient} = \frac{\sqrt{3}+\sqrt{2}}{2-\sqrt{3}} \times \frac{\sqrt{3}-\sqrt{2}}{7+4\sqrt{3}}$$
$$= \frac{(\sqrt{3})^2-(\sqrt{2})^2}{14-12+8\sqrt{3}-7\sqrt{3}}$$
$$= \frac{1}{2+\sqrt{3}} = 2-\sqrt{3}, \text{ on rationalizing.}$$

Ex. 4. Given $\sqrt{5} = 2.236068$, find the value of $\dfrac{87}{7-2\sqrt{5}}$.

Rationalizing the denominator,

$$\frac{87}{7-2\sqrt{5}} = \frac{87(7+2\sqrt{5})}{49-20}$$
$$= 3(7+2\sqrt{5})$$
$$= 34.416408.$$

It will be seen that by rationalizing the denominator we have avoided the use of a divisor consisting of 7 figures.

253. In a similar manner, where the denominator involves three quadratic surds, we may by two operations render that denominator rational.

Ex. $$\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}.$$

$$\text{The expression} = \frac{\sqrt{2}(\sqrt{2}+\sqrt{3}+\sqrt{5})}{(\sqrt{2}+\sqrt{3}-\sqrt{5})(\sqrt{2}+\sqrt{3}+\sqrt{5})}$$
$$= \frac{2+\sqrt{6}+\sqrt{10}}{2\sqrt{6}} = \frac{(2+\sqrt{6}+\sqrt{10})\sqrt{6}}{(2\sqrt{6})\sqrt{6}}$$
$$= \frac{3+\sqrt{6}+\sqrt{15}}{6}$$

EXAMPLES XXIII. g.

Find the value of

1. $(9\sqrt{2}-7)(9\sqrt{2}+7)$.

2. $(3+5\sqrt{7})(3-5\sqrt{7})$.

3. $(5\sqrt{8}-2\sqrt{7})(5\sqrt{8}+2\sqrt{7})$.

4. $(2\sqrt{11}+5\sqrt{2})(2\sqrt{11}-5\sqrt{2})$.

5. $(\sqrt{a}+2\sqrt{b})(\sqrt{a}-2\sqrt{b})$.

6. $(3c-2\sqrt{x})(3c+2\sqrt{x})$.

7. $(\sqrt{a+x}-\sqrt{a})(\sqrt{a+x}+\sqrt{a})$.

8. $(\sqrt{2p+3q}-2\sqrt{q})(\sqrt{2p+3q}+2\sqrt{q})$.

9. $(\sqrt{a+x}+\sqrt{a-x})(\sqrt{a+x}-\sqrt{a-x})$.

10. $(5\sqrt{x^2-3y^2}+7a)(5\sqrt{x^2-3y^2}-7a)$.

11. $29 \div (11 + 3\sqrt{7})$.

12. $17 \div (3\sqrt{7} + 2\sqrt{3})$.

13. $(3\sqrt{2} - 1) \div (3\sqrt{2} + 1)$.

14. $(2\sqrt{3} + 7\sqrt{2}) \div (5\sqrt{3} - 4\sqrt{2})$.

15. $(2x - \sqrt{xy}) \div (2\sqrt{xy} - y)$.

16. $(3 + \sqrt{5})(\sqrt{5} - 2) \div (5 - \sqrt{5})$.

17. $\dfrac{\sqrt{a}}{\sqrt{a} - \sqrt{x}} \div \dfrac{\sqrt{a} + \sqrt{x}}{\sqrt{x}}$.

18. $\dfrac{2\sqrt{15} + 8}{5 + \sqrt{15}} \div \dfrac{8\sqrt{3} - 6\sqrt{5}}{5\sqrt{3} - 3\sqrt{5}}$.

Rationalize the denominator of

19. $\dfrac{25\sqrt{3} - 4\sqrt{2}}{7\sqrt{3} - 5\sqrt{2}}$.

20. $\dfrac{10\sqrt{6} - 2\sqrt{7}}{3\sqrt{6} + 2\sqrt{7}}$.

21. $\dfrac{\sqrt{7} + \sqrt{2}}{9 + 2\sqrt{14}}$.

22. $\dfrac{2\sqrt{3} + 3\sqrt{2}}{5 + 2\sqrt{6}}$.

23. $\dfrac{y^2}{x + \sqrt{x^2 - y^2}}$.

24. $\dfrac{x^2}{\sqrt{x^2 + a^2} + a}$.

25. $\dfrac{\sqrt{1 + x^2} - \sqrt{1 - x^2}}{\sqrt{1 + x^2} + \sqrt{1 - x^2}}$.

26. $\dfrac{2\sqrt{a + b} + 3\sqrt{a - b}}{2\sqrt{a + b} - \sqrt{a - b}}$.

27. $\dfrac{\sqrt{9 + x^2} - 3}{\sqrt{9 + x^2} + 3}$.

28. $\dfrac{3 + \sqrt{6}}{5\sqrt{3} - 2\sqrt{12} - \sqrt{32} + \sqrt{50}}$.

29. $\dfrac{\sqrt{10} + \sqrt{5} + \sqrt{3}}{\sqrt{3} + \sqrt{10} - \sqrt{5}}$.

30. $\dfrac{(\sqrt{3} + \sqrt{5})(\sqrt{5} + \sqrt{2})}{\sqrt{2} + \sqrt{3} + \sqrt{5}}$.

Given $\sqrt{2} = 1.41421$, $\sqrt{3} = 1.73205$, $\sqrt{5} = 2.23607$: find to four places of decimals the value of

31. $\dfrac{1}{2 + \sqrt{3}}$.

32. $\dfrac{3 + \sqrt{5}}{\sqrt{5} - 2}$.

33 $\dfrac{\sqrt{5} + \sqrt{3}}{4 + \sqrt{15}}$.

34. $\dfrac{\sqrt{5} - 2}{9 - 4\sqrt{5}}$.

35. $\dfrac{7\sqrt{5} + 15}{\sqrt{5} - 1} \times \dfrac{\sqrt{5} - 2}{3 + \sqrt{5}}$.

36. $(2 - \sqrt{3})(7 - 4\sqrt{3}) \div (3\sqrt{3} - 5)$.

254. To find a factor which will rationalize any binomial surd.

Case I. Suppose the given surd is $\sqrt[p]{a} - \sqrt[q]{b}$.

Let $\sqrt[p]{a} = x$, $\sqrt[q]{b} = y$, and let n be the L. C. M. of p and q; then x^n and y^n are both rational.

Now $x^n - y^n$ is divisible by $x - y$ for all values of n, and

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1}).$$

Thus the rationalizing factor is

$$x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + y^{n-1};$$

and the rational product is $x^n - y^n$.

CASE II. Suppose the given surd is $\sqrt[p]{a} + \sqrt[q]{b}$.

Let x, y, n have the same meanings as before; then

(1) If n is even, $x^n - y^n$ is divisible by $x + y$, and

$$x^n - y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots + xy^{n-2} - y^{n-1}).$$

Thus the rationalizing factor is

$$x^{n-1} - x^{n-2}y + \cdots + xy^{n-2} - y^{n-1};$$

and the rational product is $x^n - y^n$.

(2) If n is odd, $x^n + y^n$ is divisible by $x + y$, and

$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1}).$$

Thus the rationalizing factor is

$$x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1};$$

and the rational product is $x^n + y^n$.

Ex. 1. Find the factor which will rationalize $\sqrt{3} + \sqrt[3]{5}$.

Let $x = 3^{\frac{1}{2}}$, $y = 5^{\frac{1}{3}}$; then x^6 and y^6 are both rational, and

$$x^6 - y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5);$$

thus, substituting for x and y, the required factor is

$$3^{\frac{5}{2}} - 3^{\frac{4}{2}} \cdot 5^{\frac{1}{3}} + 3^{\frac{3}{2}} \cdot 5^{\frac{2}{3}} - 3^{\frac{2}{2}} \cdot 5^{\frac{3}{3}} + 3^{\frac{1}{2}} \cdot 5^{\frac{4}{3}} - 5^{\frac{5}{3}},$$

or $$3^{\frac{5}{2}} - 9 \cdot 5^{\frac{1}{3}} + 3^{\frac{3}{2}} \cdot 5^{\frac{2}{3}} - 15 + 3^{\frac{1}{2}} \cdot 5^{\frac{4}{3}} - 5^{\frac{5}{3}};$$

and the rational product is $3^{\frac{6}{2}} - 5^{\frac{6}{3}} = 3^3 - 5^2 = 2$.

Ex. 2. Express $(5^{\frac{1}{2}} + 9^{\frac{1}{8}}) \div (5^{\frac{1}{2}} - 9^{\frac{1}{8}})$

as an equivalent fraction with a rational denominator.

To rationalize the denominator, which is equal to $5^{\frac{1}{2}} - 3^{\frac{1}{4}}$, put $5^{\frac{1}{2}} = x$, $3^{\frac{1}{4}} = y$; then since

$$x^4 - y^4 = (x - y)(x^3 + x^2y + xy^2 + y^3),$$

the required factor is $5^{\frac{3}{2}} + 5^{\frac{2}{2}} \cdot 3^{\frac{1}{4}} + 5^{\frac{1}{2}} \cdot 3^{\frac{2}{4}} + 3^{\frac{3}{4}}$;

and the rational denominator is $5^{\frac{4}{2}} - 3^{\frac{4}{4}} = 5^2 - 3 = 22$.

$$\therefore \text{ the expression } = \frac{(5^{\frac{1}{2}} + 3^{\frac{1}{4}})(5^{\frac{3}{2}} + 5^{\frac{2}{2}} \cdot 3^{\frac{1}{4}} + 5^{\frac{1}{2}} \cdot 3^{\frac{2}{4}} + 3^{\frac{3}{4}})}{22}$$

$$= \frac{5^{\frac{4}{2}} + 2 \cdot 5^{\frac{3}{2}} \cdot 3^{\frac{1}{4}} + 2 \cdot 5^{\frac{2}{2}} \cdot 3^{\frac{2}{4}} + 2 \cdot 5^{\frac{1}{2}} \cdot 3^{\frac{3}{4}} + 3^{\frac{4}{4}}}{22}$$

$$= \frac{14 + 5^{\frac{3}{2}} \cdot 3^{\frac{1}{4}} + 5 \cdot 3^{\frac{1}{2}} + 5^{\frac{1}{2}} \cdot 3^{\frac{3}{4}}}{11}$$

PROPERTIES OF QUADRATIC SURDS.

255. The square root of a rational quantity cannot be partly rational and partly a quadratic surd.

If possible let $\sqrt{n} = a + \sqrt{m}$;

then by squaring, $n = a^2 + m + 2a\sqrt{m}$;

$$\therefore \sqrt{m} = \frac{n - a^2 - m}{2a},$$

that is, a surd is equal to a rational quantity, which is impossible.

256. If $x + \sqrt{y} = a + \sqrt{b}$, then will $x = a$ and $y = b$.

For if x is not equal to a, let $x = a + m$; then

$$a + m + \sqrt{y} = a + \sqrt{b};$$

that is, $\sqrt{b} = m + \sqrt{y}$;

which is impossible. [Art. 255.]

Therefore $x = a$,

and consequently, $y = b$.

If therefore $x + \sqrt{y} = a + \sqrt{b}$,

we must also have $x - \sqrt{y} = a - \sqrt{b}$.

257. It appears from the preceding article that in any equation of the form

$$x + \sqrt{y} = a + \sqrt{b} \quad . \quad . \quad . \quad . \quad . \quad (1),$$

we may equate the rational parts on each side, and also the irrational parts; so that the equation (1) is really equivalent to *two* independent equations, $x = a$ and $y = b$.

258. If $\sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y}$, then will $\sqrt{a - \sqrt{b}}$ $= \sqrt{x} - \sqrt{y}$.

For by squaring, we obtain

$$a + \sqrt{b} = x + 2\sqrt{xy} + y;$$

$$\therefore a = x + y, \quad \sqrt{b} = 2\sqrt{xy}. \quad \text{[Art. 257.]}$$

Hence $a - \sqrt{b} = x - 2\sqrt{xy} + y$,

and $\sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}$.

259. To find the square root of a binomial surd.

Suppose $\sqrt{a+\sqrt{b}}=\sqrt{x}+\sqrt{y}$;

then as in Art. 258,

$$x+y=a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1);$$

$$2\sqrt{xy}=\sqrt{b} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Since
$$(x-y)^2=(x+y)^2-4xy$$
$$=a^2-b \quad . \quad . \quad \text{from (1) and (2)};$$
$$\therefore\ x-y=\sqrt{a^2-b}.$$

Combining this with (1), we find

$$x=\frac{a+\sqrt{a^2-b}}{2}, \text{ and } y=\frac{a-\sqrt{a^2-b}}{2}.$$

$$\therefore\ \sqrt{a+\sqrt{b}}=\sqrt{\frac{a+\sqrt{(a^2-b)}}{2}}+\sqrt{\frac{a-\sqrt{(a^2-b)}}{2}}.$$

260. The values just found for x and y are compound surds unless a^2-b is a perfect square. Hence the method of Art. 259 for finding the square root of $a+\sqrt{b}$ is of no practical utility except when a^2-b is a perfect square.

Ex. Find the square root of $16+2\sqrt{55}$.

Assume $\sqrt{16+2\sqrt{55}}=\sqrt{x}+\sqrt{y}$.

Then $16+2\sqrt{55}=x+2\sqrt{xy}+y$;

$$\therefore\ x+y=16 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

$$2\sqrt{xy}=2\sqrt{55} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Since
$$(x-y)^2=(x+y)^2-4xy$$
$$=16^2-4\times 55 \quad . \quad . \quad . \quad \text{by (1) and (2)},$$
$$=4\times 9.$$
$$\therefore\ x-y=\pm 6 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

From (1) and (3) we obtain

$$x=11, \text{ or } 5, \text{ and } y=5, \text{ or } 11.$$

That is, the required square root is $\sqrt{11}+\sqrt{5}$.

In the same way we may show that

$$\sqrt{16-2\sqrt{55}}=\sqrt{11}-\sqrt{5}.$$

NOTE. Since every quantity has two square roots equal in magnitude but opposite in sign, strictly speaking we should have

$$\text{the square root of } 16+2\sqrt{55}=\pm(\sqrt{11}+\sqrt{5}),$$

$$\text{the square root of } 16-2\sqrt{55}=\pm(\sqrt{11}-\sqrt{5}).$$

However, it is usually sufficient to take the positive value of the square root, so that in assuming $\sqrt{a-\sqrt{b}}=\sqrt{x}-\sqrt{y}$ it is understood that x is greater than y. With this proviso it will be unnecessary in any numerical example to use the double sign at the stage of work corresponding to equation (3) of the last example.

261. When the binomial whose square root we are seeking consists of *two* quadratic surds, we proceed as explained in the following example.

Ex. Find the square root of $\sqrt{175}-\sqrt{147}$.

Since $\sqrt{175}-\sqrt{147}=\sqrt{7}(\sqrt{25}-\sqrt{21})=\sqrt{7}(5-\sqrt{21})$,

$$\therefore \sqrt{\sqrt{175}-\sqrt{147}}=\sqrt[4]{7}\cdot\sqrt{5-21}.$$

But

$$\sqrt{5-\sqrt{21}}=\sqrt{\tfrac{7}{2}}-\sqrt{\tfrac{3}{2}};$$

$$\therefore \sqrt{\sqrt{175}-\sqrt{147}}=\sqrt[4]{7}\left(\sqrt{\tfrac{7}{2}}-\sqrt{\tfrac{3}{2}}\right).$$

262. To find the square root of a binomial surd by inspection.

Ex. 1. Find the square root of $11+2\sqrt{30}$.

We have only to find two quantities whose sum is 11, and whose product is 30, thus

$$11+2\sqrt{30}=6+5+2\sqrt{6\times5}=(\sqrt{6}+\sqrt{5})^2.$$

$$\therefore \sqrt{11+2\sqrt{30}}=\sqrt{6}+\sqrt{5}.$$

Ex. 2. Find the square root of $53-12\sqrt{10}$.

First write the binomial so that the surd part has a coefficient 2; thus

$$53-12\sqrt{10}=53-2\sqrt{360}.$$

We have now to find two quantities whose sum is 53 and whose product is 360; these are 45 and 8;

hence

$$\begin{aligned}53-12\sqrt{10}&=45+8-2\sqrt{45\times8}\\&=(\sqrt{45}-\sqrt{8})^2;\end{aligned}$$

$$\begin{aligned}\therefore \sqrt{53-12\sqrt{10}}&=\sqrt{45}-\sqrt{8}\\&=3\sqrt{5}-2\sqrt{2}.\end{aligned}$$

Ex. 3. Find the square root of $a + b + \sqrt{2ab + b^2}$.

Rewrite the binomial so that the surd part has a coefficient 2; thus

$$a + b + \sqrt{2ab + b^2} = a + b + 2\sqrt{\frac{2ab + b^2}{4}}.$$

We have now to find two quantities whose sum is $(a + b)$ and whose product is $\frac{2ab + b^2}{4}$; these are $\frac{2a + b}{2}$, and $\frac{b}{2}$; hence

$$a + b + \sqrt{2ab + b^2} = \frac{2a + b}{2} + \frac{b}{2} + 2\sqrt{\frac{2a + b}{2} \times \frac{b}{2}}$$

$$= \left(\sqrt{\frac{2a + b}{2}} + \sqrt{\frac{b}{2}}\right)^2;$$

$$\therefore \sqrt{a + b + \sqrt{2ab + b^2}} = \sqrt{a + \frac{b}{2}} + \sqrt{\frac{b}{2}}.$$

Note. The student should observe that when the coefficient of the surd part of the binomial is unity, he can make this coefficient 2 if he will also *multiply the quantity under the radical by* $\frac{1}{4}$.

263. Assuming $\sqrt[3]{a + \sqrt{b}} = x + \sqrt{y}$, the method of Art. 258 gives us $\sqrt[3]{a - \sqrt{b}} = x - \sqrt{y}$.

EXAMPLES XXIII. h.

Find the square roots of the following binomial surds:

1. $7 - 2\sqrt{10}$.
2. $13 + 2\sqrt{30}$.
3. $8 - 2\sqrt{7}$.
4. $5 + 2\sqrt{6}$.
5. $75 + 12\sqrt{21}$.
6. $18 - 8\sqrt{5}$.
7. $41 - 24\sqrt{2}$.
8. $83 + 12\sqrt{35}$.
9. $47 - 4\sqrt{33}$.
10. $2\frac{1}{4} + \sqrt{5}$.
11. $4\frac{1}{3} - \frac{4}{3}\sqrt{3}$.
12. $16 + 5\sqrt{7}$.
13. $\sqrt{27} + 2\sqrt{6}$.
14. $\sqrt{32} - \sqrt{24}$.
15. $3\sqrt{5} + \sqrt{40}$.
16. $2a + 2\sqrt{a^2 - b^2}$.
17. $ax - 2a\sqrt{ax - a^2}$.
18. $a + x + \sqrt{2ax + x^2}$.
19. $2a - \sqrt{3a^2 - 2ab - b^2}$.
20. $1 + a^2 + (1 + a^2 + a^4)^{\frac{1}{2}}$

Find the fourth roots of the following binomial surds:

21. $17 + 12\sqrt{2}$.
22. $56 + 24\sqrt{5}$.
23. $\frac{3}{2}\sqrt{5} + 3\frac{1}{2}$.
24. $14 + 8\sqrt{3}$.
25. $49 - 20\sqrt{6}$.
26. $248 + 32\sqrt{60}$.

Find, by inspection, the value of

27. $\sqrt{3 - 2\sqrt{2}}$.
28. $\sqrt{4 + 2\sqrt{3}}$.
29. $\sqrt{6 - 2\sqrt{5}}$.
30. $\sqrt{19 + 8\sqrt{3}}$.
31. $\sqrt{8 + 2\sqrt{15}}$.
32. $\sqrt{9 - 2\sqrt{14}}$.
33. $\sqrt{11 + 4\sqrt{6}}$.
34. $\sqrt{15 - 4\sqrt{14}}$.
35. $\sqrt{29 + 6\sqrt{22}}$.

Express with rational denominator

36. $\dfrac{1}{\sqrt[3]{3}-\sqrt{2}}$.

38. $\dfrac{\sqrt{2}\cdot\sqrt[3]{3}}{\sqrt[3]{3}+\sqrt{2}}$.

40. $\dfrac{1}{\sqrt[3]{5}-\sqrt[4]{3}}$.

37. $\dfrac{1}{\sqrt[3]{3}-1}$.

39. $\dfrac{\sqrt{8}+\sqrt[3]{4}}{\sqrt{8}-\sqrt[3]{4}}$.

41. $\dfrac{\sqrt[3]{3}}{\sqrt{3}+\sqrt[6]{9}}$.

42. $\dfrac{1}{2+\sqrt[4]{7}}$.

43. Find value of $\sqrt{\dfrac{6+2\sqrt{3}}{33-19\sqrt{3}}}$.

EQUATIONS INVOLVING SURDS.

264. Sometimes equations are proposed in which the unknown quantity appears under the radical sign. Such equations are varied in character and often require special artifices for their solution. We shall consider a few of the simpler cases, which can generally be solved by the following method:

Bring to one side of the equation a single radical term by itself: on squaring both sides this radical will disappear. By repeating this process any remaining radicals can in turn be removed.

Ex. 1. Solve $2\sqrt{x}-\sqrt{4x-11}=1$.

Transposing, $2\sqrt{x}-1=\sqrt{4x-11}$.

Square both sides; then $4x-4\sqrt{x}+1=4x-11$,

$$4\sqrt{x}=12,$$
$$\sqrt{x}=3;$$
$$\therefore\ x=9.$$

Ex. 2. Solve $2+\sqrt[3]{x-5}=13$.

Transposing, $\sqrt[3]{x-5}=11$.

Here we must *cube* both sides; thus $x-5=1331$;

whence $x=1336$.

EXAMPLES XXIII. k.

1. $\sqrt{x-5}=3$.

2. $\sqrt[3]{4x-7}=5$.

3. $7-\sqrt{x-4}=3$.

4. $13-\sqrt[3]{5x-4}=7$.

5. $\sqrt{5x-1}=2\sqrt{x+3}$.

6. $2\sqrt{3-7x}-3\sqrt{8x-12}=0$.

7. $2\sqrt[3]{5x-35}=5\sqrt[3]{2x-7}$.

8. $\sqrt{9x^2-11x-5}=3x-2$.

9. $\sqrt[4]{2x+11}=\sqrt{5}$.

10. $\sqrt{4x^2-7x+1}=2x-1\frac{4}{5}$.

11. $\sqrt{x+25}=1+\sqrt{x}$.

12. $\sqrt{8x+33}-3=2\sqrt{2x}$.

13. $\sqrt{x+3}+\sqrt{x}=5$.

14. $10-\sqrt{25+9x}=3\sqrt{x}$.

15. $\sqrt{x-4}+3=\sqrt{x+11}$.

16. $\sqrt{9x-8}=3\sqrt{x+4}-2$.

17. $\sqrt{4x+5}-\sqrt{x}=\sqrt{x+3}$.

18. $\sqrt{25x-29}-\sqrt{4x-11}=3\sqrt{x}$.

19. $\sqrt{x+4ab}=2a+\sqrt{x}$.

20. $\sqrt{x}+\sqrt{4a+x}=2\sqrt{b+x}$.

265. When radicals appear in a fractional form in an equation, we must clear of fractions in the ordinary way, combining the irrational factors by the rules already explained in this chapter.

Ex. Solve $\quad \sqrt{9+2x}-\sqrt{2x}=\dfrac{5}{\sqrt{9+2x}}$.

Clearing of fractions,

$$9+2x-\sqrt{2x(9+2x)}=5,$$

$$4+2x=\sqrt{2x(9+2x)}.$$

Squaring, $\quad 16+16x+4x^2=18x+4x^2,$

$$16=2x,$$

$$x=8.$$

EXAMPLES XXIII. 1.

1. $\dfrac{6\sqrt{x}-21}{3\sqrt{x}-14}=\dfrac{8\sqrt{x}-11}{4\sqrt{x}-13}$.

2. $\dfrac{9\sqrt{x}-23}{3\sqrt{x}-8}=\dfrac{6\sqrt{x}-17}{2\sqrt{x}-6}$.

3. $\dfrac{\sqrt{x}+3}{\sqrt{x}-2}=\dfrac{3\sqrt{x}-5}{3\sqrt{x}-13}$.

4. $2-\dfrac{\sqrt{x}+3}{\sqrt{x}+2}=\dfrac{\sqrt{x}+9}{\sqrt{x}+7}$.

5. $\dfrac{2\sqrt{x}-1}{2\sqrt{x}+\frac{4}{3}}=\dfrac{\sqrt{x}-2}{\sqrt{x}-\frac{4}{3}}$.

6. $\dfrac{6\sqrt{x}-7}{\sqrt{x}-1}-5=\dfrac{7\sqrt{x}-26}{7\sqrt{x}-21}$.

7. $\dfrac{12\sqrt{x}-11}{4\sqrt{x}-4\frac{2}{3}}=\dfrac{6\sqrt{x}+5}{2\sqrt{x}+\frac{2}{3}}$.

8. $\sqrt{1+x}+\sqrt{x}=\dfrac{2}{\sqrt{1+x}}$.

9. $\sqrt{x-1}+\sqrt{x}=\dfrac{2}{\sqrt{x}}$.

10. $\sqrt{x}-\sqrt{x-8}=\dfrac{2}{\sqrt{x-8}}$.

11. $\sqrt{x+5}+\sqrt{x}=\dfrac{10}{\sqrt{x}}$.

12. $2\sqrt{x}-\sqrt{4x-3}=\dfrac{1}{\sqrt{4x-3}}$.

13. $3\sqrt{x}=\dfrac{8}{\sqrt{9x-32}}+\sqrt{9x-32}$.

14. $\sqrt{x}-7=\dfrac{1}{\sqrt{x}+7}$.

15. $(\sqrt{x}+11)(\sqrt{x}-11)+110=0$.

16. $2\sqrt{x}=\dfrac{12-6\sqrt{x}}{2\sqrt{x}-3}$.

CHAPTER XXIV.

IMAGINARY QUANTITIES.

266. An **imaginary quantity** is an indicated even root of a negative quantity. In distinction from imaginary quantities all other quantities are spoken of as **real quantities.** Although from the rule of signs it is evident that a negative quantity cannot have a real square root, yet quantities represented by symbols of the form $\sqrt{-a}$, $\sqrt{-1}$, are of frequent occurrence in mathematical investigations, and their use leads to valuable results. We therefore proceed to explain in what sense such roots are to be regarded.

When the quantity under the radical sign is negative, we can no longer consider the symbol $\sqrt{}$ as indicating a possible arithmetical operation; but just as $\sqrt{a}$ may be defined as a symbol which obeys the relation $\sqrt{a} \times \sqrt{a} = a$, so we shall define $\sqrt{-a}$ to be such that $\sqrt{-a} \times \sqrt{-a} = -a$, and we shall accept the meaning to which this assumption leads us.

It will be found that this definition will enable us to bring imaginary quantities under the dominion of ordinary algebraic rules, and that through their use results may be obtained which can be relied on with as much certainty as others which depend solely on the use of real quantities.

267. Any imaginary expression not involving the operation of raising to a power indicated by an exponent that is an irrational or imaginary expression, can be reduced to the form $a + b\sqrt{-1}$, which may be taken as the *general type* of all imaginary expressions. Here a and b are real quantities, but not necessarily rational. An imaginary expression in this form is called a **complex number.** If $a=0$, the form becomes $b\sqrt{-1}$, which is called a **pure imaginary expression.**

268. By definition, $\sqrt{-1}\times\sqrt{-1}=-1.$

$$\therefore\ \sqrt{a}\cdot\sqrt{-1}\times\sqrt{a}\cdot\sqrt{-1}=a(-1);$$

that is,
$$(\sqrt{a}\cdot\sqrt{-1})^2=-a.$$

Thus the product $\sqrt{a}\,.\,\sqrt{-1}$ may be regarded as equivalent to the imaginary quantity $\sqrt{-a}$.

269. It will generally be found convenient to indicate the imaginary character of an expression by the presence of the symbol $\sqrt{-1}$ which is called the **imaginary unit**; thus

$$\sqrt{-4}=\sqrt{4\times(-1)}=2\sqrt{-1}.$$
$$\sqrt{-7a^2}=\sqrt{7a^2\times(-1)}=a\sqrt{7}\sqrt{-1}.$$

270. We shall always consider that, in the absence of any statement to the contrary, of the signs which may be prefixed before a radical the positive sign is to be taken. But in the use of imaginary quantities the following point deserves notice.

Since
$$(-a)\times(-b)=ab,$$
by taking the square root, we have

$$\sqrt{-a}\times\sqrt{-b}=\pm\sqrt{ab}.$$

Thus in forming the product of $\sqrt{-a}$ and $\sqrt{-b}$ it would appear that either of the signs $+$ or $-$ might be placed before $\sqrt{ab}$. This is not the case, for

$$\begin{aligned}\sqrt{-a}\times\sqrt{-b}&=\sqrt{a}\,.\,\sqrt{-1}\times\sqrt{b}\cdot\sqrt{-1}\\&=\sqrt{ab}(\sqrt{-1})^2=-\sqrt{ab}.\end{aligned}$$

271. In dealing with imaginary quantities we apply the laws of combination which have been proved in the case of other surd quantities.

Ex. 1. $a+b\sqrt{-1}\pm(c+d\sqrt{-1})=a\pm c+(b\pm d)\sqrt{-1}.$

Ex. 2. The product of $a+b\sqrt{-1}$ and $c+d\sqrt{-1}$

$$\begin{aligned}&=(a+b\sqrt{-1})(c+d\sqrt{-1})\\&=ac-bd+(bc+ad)\sqrt{-1}.\end{aligned}$$

272. The symbol $\sqrt{-1}$ is often represented by the letter i; but until the student has had a little practice in the use of imaginary quantities he will find it easier to retain the symbol $\sqrt{-1}$. The successive powers of $\sqrt{-1}$, or i, are as follows:

$$\begin{aligned}
(\sqrt{-1})^1 &= \sqrt{-1}, & i &= i;\\
(\sqrt{-1})^2 &= -1, & i^2 &= -1;\\
(\sqrt{-1})^3 &= -\sqrt{-1}, & i^3 &= -i;\\
(\sqrt{-1})^4 &= 1, & i^4 &= 1;
\end{aligned}$$

and since each power is obtained by multiplying the one before it by $\sqrt{-1}$, or i, we see that the results must now recur.

273. If $a + b\sqrt{-1} = 0$, then $a = 0$, and $b = 0$.

For, if $$a + b\sqrt{-1} = 0,$$

then $$b\sqrt{-1} = -a;$$
$$\therefore\ -b^2 = a^2;$$
$$\therefore\ a^2 + b^2 = 0.$$

Now a^2 and b^2 are both positive, hence their sum cannot be zero unless each is separately zero; that is, $a = 0$, and $b = 0$.

274. If $a + b\sqrt{-1} = c + d\sqrt{-1}$, then $a = c$, and $b = d$.

For, by transposition, $a - c + (b - d)\sqrt{-1} = 0$;

therefore, by the last article, $a - c = 0$, and $b - d = 0$;

that is, $$a = c \text{ and } b = d.$$

Thus in order that two imaginary expressions may be equal it is necessary and sufficient *that the real parts should be equal, and the imaginary parts should be equal.*

The student should carefully note this article and make use of it as opportunity may offer in the solution of equations involving imaginary expressions.

275. When two imaginary expressions differ only in the sign of the imaginary part, they are said to be **conjugate.**

Thus $a - b\sqrt{-1}$ is conjugate to $a + b\sqrt{-1}$.

Similarly $\sqrt{2} + 3\sqrt{-1}$ is conjugate to $\sqrt{2} - 3\sqrt{-1}$.

276. The sum and the product of two conjugate imaginary expressions are both real.

For $\qquad a + b\sqrt{-1} + a - b\sqrt{-1} = 2a.$

Again $(a + b\sqrt{-1})(a - b\sqrt{-1}) = a^2 - (-b^2) = a^2 + b^2.$

277. If the denominator of a fraction is of the form $a + b\sqrt{-1}$, it may be rationalized by multiplying the numerator and the denominator by the conjugate expression $a - b\sqrt{-1}$. For instance,

$$\frac{c + d\sqrt{-1}}{a + b\sqrt{-1}} = \frac{(c + d\sqrt{-1})(a - b\sqrt{-1})}{(a + b\sqrt{-1})(a - b\sqrt{-1})}$$

$$= \frac{ac + bd + (ad - bc)\sqrt{-1}}{a^2 + b^2}$$

$$= \frac{ac + bd}{a^2 + b^2} + \frac{ad - bc}{a^2 + b^2}\sqrt{-1}.$$

Thus, by reference to Art. 271, we see that *the sum, difference, product, and quotient of two imaginary expressions is in each case an imaginary expression of the same form.*

278. Fundamental Algebraic Operations upon Imaginary Quantities.

Ex. 1. Find value of $\sqrt{-a^4} + 5\sqrt{-9a^4} - 2\sqrt{-4a^4}$.

$$\begin{aligned} \sqrt{-a^4} &= \sqrt{a^4(-1)} &&= a^2\sqrt{-1} \\ 5\sqrt{-9a^4} &= 5\sqrt{9a^4(-1)} &&= 15a^2\sqrt{-1} \\ -2\sqrt{-4a^4} &= -2\sqrt{4a^4(-1)} &&= -4a^2\sqrt{-1} \\ \hline & &&= 12a^2\sqrt{-1} \end{aligned}$$

Ex. 2. Multiply $2\sqrt{-3}$ by $3\sqrt{-2}$.

$$2\sqrt{-3} = 2\sqrt{3}\sqrt{-1};$$

$$3\sqrt{-2} = 3\sqrt{2}\sqrt{-1};$$

$$(2\sqrt{3}\sqrt{-1})(3\sqrt{2}\sqrt{-1}) = 6\sqrt{6}(\sqrt{-1})^2 = -6\sqrt{6}.$$

Ex. 3. Divide $2+3\sqrt{-1}$ by $2+\sqrt{-1}$.

$$\frac{2+3\sqrt{-1}}{2+\sqrt{-1}}=\frac{(2+3\sqrt{-1})(2-\sqrt{-1})}{(2+\sqrt{-1})(2-\sqrt{-1})}=\frac{7+4\sqrt{-1}}{4-(-1)}=\frac{7+4\sqrt{-1}}{5}.$$

279. The method of Art. 262 may be used in finding **the square root of $a+b\sqrt{-1}$.**

Ex. Find the square root of $-7-24\sqrt{-1}$.

$$-7-24\sqrt{-1}=-7-2\sqrt{-144}.$$

We have now to find two quantities whose sum is -7 and whose product is -144; these are 9 and -16;

hence
$$-7-24\sqrt{-1}=9+(-16)-2\sqrt{9\times(-16)}$$
$$=(\sqrt{9}-\sqrt{-16})^2;$$
$$\therefore \sqrt{-7-24\sqrt{-1}}=\pm(3-4\sqrt{-1}).$$

EXAMPLES XXIV.

Simplify:

1. $\sqrt{-8}+\sqrt{-18}$.
2. $4\sqrt{-27}+3\sqrt{-12}$.
3. $5\sqrt{-16}-2\sqrt{-9}$.
4. $2\sqrt{-20}+3\sqrt{-45}-\sqrt{-80}$.
5. $2\sqrt{-a^2x^2}+7\sqrt{-4a^2x^2}+12\sqrt{-36a^2x^2}$.
6. $\sqrt{-\frac{1}{2}}-\sqrt{-\frac{9}{4}}+\sqrt{-\frac{27}{16}}+\sqrt{-\frac{1}{8}}$.
7. $(\sqrt{-3})(\sqrt{-12})$.
8. $(2+\sqrt{-2})(1-\sqrt{-3})$.
9. $(2\sqrt{-2}+\sqrt{-3})(\sqrt{-3}-\sqrt{-5})$.
10. $(2+\sqrt{-a})(3-\sqrt{-a})$.
11. $(4+\sqrt{-2})(2-3\sqrt{-5})$.
12. $(2\sqrt{-3}+3\sqrt{-2})(4\sqrt{-3}-5\sqrt{-2})$.
13. $\sqrt{27}\div\sqrt{-3}$.
14. $-\sqrt{-4}\div(\sqrt{-2}+\sqrt{-3})$.
15. $(-\sqrt{-a^2}+\sqrt{-2})\div(2\sqrt{-1}-\sqrt{-2})$.

Express with rational denominator:

16. $\dfrac{1}{3-\sqrt{-2}}$.
17. $\dfrac{3\sqrt{-2}+2\sqrt{-5}}{3\sqrt{-2}-2\sqrt{-5}}$.
18. $\dfrac{3+2\sqrt{-1}}{2-5\sqrt{-1}}+\dfrac{3-2\sqrt{-1}}{2+5\sqrt{-1}}$.
19. $\dfrac{a+x\sqrt{-1}}{a-x\sqrt{-1}}-\dfrac{a-x\sqrt{-1}}{a+x\sqrt{-1}}$.

Find the square root of

20. $-5+12\sqrt{-1}$.
21. $-11-60\sqrt{-1}$.
22. $-47+8\sqrt{-3}$.

Express in the form $a+ib$:

23. $\dfrac{3+5i}{2-3i}$.
24. $\dfrac{\sqrt{3}-i\sqrt{2}}{2\sqrt{3}-i\sqrt{2}}$.
25. $\dfrac{1+i}{1-i}$.

CHAPTER XXV.

PROBLEMS.

280. In previous chapters we have given collections of problems which lead to simple equations. We add here a few examples of somewhat greater difficulty.

Ex. **1.** A grocer buys 15 lbs. of figs and 28 lbs. of currants for \$2.60; by selling the figs at a loss of 10 per cent, and the currants at a gain of 30 per cent, he clears 30 cents on his outlay: how much per pound did he pay for each?

Let x, y denote the number of cents in the price of a pound of figs and currants respectively; then the outlay is

$$15x + 28y \text{ cents.}$$

$$\therefore 15x + 28y = 260 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The loss upon the figs is $\frac{1}{10} \times 15x$ cents, and the gain upon the currants is $\frac{3}{10} \times 28y$ cents; therefore the total gain is

$$\frac{42y}{5} - \frac{3x}{2} \text{ cents;}$$

$$\therefore \frac{42y}{5} - \frac{3x}{2} = 30 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

From (1) and (2) we find that $x = 8$, and $y = 5$; that is, the figs cost 8 cents a pound, and the currants cost 5 cents a pound.

Ex. **2.** At what time between 4 and 5 o'clock will the minute-hand of a watch be 13 minutes in advance of the hour-hand?

Let x denote the required number of minutes after 4 o'clock; then, as the minute-hand travels twelve times as fast as the hour-hand, the hour-hand will move over $\frac{x}{12}$ minute divisions in x minutes. At 4 o'clock the minute-hand is 20 divisions behind the hour-hand, and finally is 13 divisions in advance; therefore the minute-hand moves over $20 + 13$, or 33 divisions more than the hour-hand.

Hence $$x = \frac{x}{12} + 33,$$

$$\tfrac{11}{12}x = 33;$$

$$\therefore\ x = 36.$$

Thus the time is 36 minutes past 4.

If the question be asked as follows: "At what *times* between 4 and 5 o'clock will there be 13 minutes between the two hands?" we must also take into consideration the case when the minute-hand is 13 divisions *behind* the hour-hand. In this case the minute-hand gains $20 - 13$, or 7 divisions.

Hence $$x = \frac{x}{12} + 7,$$

which gives $$x = 7\tfrac{7}{11}.$$

Therefore the *times* are $7\frac{7'}{11}$ past 4, and $36'$ past 4.

Ex. 3. Two persons A and B start simultaneously from two places, c miles apart, and walk in the same direction. A travels at the rate of p miles an hour, and B at the rate of q miles; how far will A have walked before he overtakes B?

Suppose A has walked x miles, then B has walked $x - c$ miles.

A, walking at the rate of p miles an hour, will travel x miles in $\frac{x}{p}$ hours; and B will travel $x - c$ miles in $\frac{x-c}{q}$ hours: these two times being equal, we have

$$\frac{x}{p} = \frac{x-c}{q},$$

$$qx = px - pc;$$

whence $$x = \frac{pc}{p-q}.$$

Therefore A has travelled $\frac{pc}{p-q}$ miles.

Ex. 4. A train travelled a certain distance at a uniform rate. Had the speed been 6 miles an hour more, the journey would have occupied 4 hours less; and had the speed been 6 miles an hour less, the journey would have occupied 6 hours more. Find the distance.

Let the speed of the train be x miles per hour, and let the time occupied be y hours; then the distance traversed will be represented by xy miles.

On the first supposition the speed per hour is $x+6$ miles, and the time taken is $y-4$ hours. In this case the distance traversed will be represented by $(x+6)(y-4)$ miles.

On the second supposition the distance traversed will be represented by $(x-6)(y+6)$ miles.

All these expressions for the distance must be equal;

$$\therefore\ xy=(x+6)(y-4)=(x-6)(y+6).$$

From these equations we have

$$xy = xy + 6y - 4x - 24,$$

or $$6y - 4x = 24 \qquad \ldots \qquad (1);$$

and $$xy = xy - 6y + 6x - 36,$$

or $$6x - 6y = 36 \qquad \ldots \qquad (2).$$

From (1) and (2) we obtain $x=30$, $y=24$.

Hence the distance is 720 miles.

Ex. 5. A person invests \$3770, partly in 3 per cent Bonds at \$102, and partly in Railway Stock at \$84 which pays a dividend of $4\frac{1}{2}$ per cent; if his income from these investments is \$136.25 per annum, what sum does he invest in each?

Let x denote the number of dollars invested in Bonds, y the number of dollars invested in Railway Stock; then

$$x+y=3770 \qquad \ldots \qquad (1).$$

The income from Bonds is $\$\frac{3x}{102}$, or $\$\frac{x}{34}$; and that from Railway Stock is $\$\frac{4\frac{1}{2}y}{84}$, or $\$\frac{3y}{56}$.

Therefore $$\frac{x}{34}+\frac{3y}{56}=136\tfrac{1}{4} \qquad \ldots \qquad (2).$$

From (2) $$x+\tfrac{51}{28}y=4632\tfrac{1}{2},$$

and by subtracting (1) $$\tfrac{23}{28}y=862\tfrac{1}{2};$$

whence $$y=28\times 37\tfrac{1}{2}=1050;$$

and from (1) $$x=2720.$$

Therefore he invests \$2720 in Bonds and \$1050 in Railway Stock.

EXAMPLES XXV.

1. A sum of \$100 is divided among a number of persons; if the number had been increased by one-fourth each would have received a half-dollar less: find the number of persons.

2. I bought a certain number of marbles at four for a cent; I kept one-fifth of them, and sold the rest at three for a cent, and gained a cent: how many did I buy?

3. I bought a certain number of articles at five for six cents; if they had been eleven for twelve cents, I should have spent six cents less: how many did I buy?

4. A man at whist wins twice as much as he had to begin with, and then loses $16; he then loses four-fifths of what remained, and afterwards wins as much as he had at first: how much had he originally, if he leaves off with $80?

5. A number of two digits exceeds five times the sum of its digits by 9, and its ten-digit exceeds its unit-digit by 1: find it.

6. The sum of the digits of a number less than 100 is 6; if the digits be reversed the resulting number will be less by 18 than the original number: find it.

7. A man being asked his age replied, "If you take 2 years from my present age the result will be double my wife's age, and 3 years ago her age was one-third of what mine will be in 12 years." What were their ages?

8. At what time between one and two o'clock are the hands of a watch first at right angles?

9. At what time between 3 and 4 o'clock is the minute-hand one minute ahead of the hour-hand?

10. When are the hands of a clock together between the hours of 6 and 7?

11. It is between 2 and 3 o'clock, and in 10 minutes the minute-hand will be as much before the hour-hand as it is now behind it: what is the time?

12. At an election a majority of 162 was three-elevenths of the whole number of voters: find the number of votes on each side.

13. A certain number of persons paid a bill; if there had been 10 more each would have paid $2 less; if there had been 5 less each would have paid $2.50 more: find the number of persons, and what each had to pay.

14. A man spends $100 in buying two kinds of silk at $4.50 and $4 a yard; by selling it at $4.25 per yard he gains 2 per cent: how much of each did he buy?

15. Ten years ago the sum of the ages of two sons was one-third of their father's age: one is two years older than the other, and the present sum of their ages is fourteen years less than their father's age: how old are they?

16. A basket of oranges is emptied by one person taking half of them and one more, a second person taking half of the remainder and one more, and a third person taking half of the remainder and six more. How many did the basket contain at first?

17. A person swimming in a stream which runs $1\frac{1}{2}$ miles per hour, finds that it takes him four times as long to swim a mile up the stream as it does to swim the same distance down: at what rate does he swim?

18. At what *times* between 7 and 8 o'clock will the hands of a watch be at right angles to each other? When will they be in the same straight line?

19. The denominator of a fraction exceeds the numerator by 4; and if 5 is taken from each, the sum of the reciprocal of the new fraction and four times the original fraction is 5: find the original fraction.

20. Two persons start at noon from towns 60 miles apart. One walks at the rate of four miles an hour, but stops $2\frac{1}{2}$ hours on the way; the other walks at the rate of 3 miles an hour without stopping: when and where will they meet?

21. A, B, and C travel from the same place at the rates of 4, 5, and 6 miles an hour respectively; and B starts 2 hours after A. How long after B must C start in order that they may overtake A at the same instant?

22. A dealer bought a horse, expecting to sell it again at a price that would have given him 10 per cent profit on his purchase; but he had to sell it for $50 less than he expected, and he then found that he had lost 15 per cent on what it cost him: what did he pay for the horse?

23. A man walking from a town, A, to another, B, at the rate of 4 miles an hour, starts one hour before a coach travelling 12 miles an hour, and is picked up by the coach. On arriving at B, he finds that his coach journey has lasted 2 hours: find the distance between A and B.

24. What is the property of a person whose income is $1140, when one-twelfth of it is invested at 2 per cent, one-half at 3 per cent, one-third at $4\frac{1}{2}$ per cent, and the remainder pays him no dividend?

25. A person spends one-third of his income, saves one-fourth, and pays away 5 per cent on the whole as interest at $7\frac{1}{2}$ per cent on debts previously incurred, and then has $110 remaining: what was the amount of his debts?

26. Two vessels contain mixtures of wine and water; in one there is three times as much wine as water, in the other five times as much water as wine. Find how much must be drawn off from each to fill a third vessel which holds seven gallons, in order that its contents may be half wine and half water.

27. There are two mixtures of wine and water, one of which contains twice as much water as wine, and the other three times as much wine as water. How much must there be taken from each to fill a pint cup, in which the water and wine shall be equally mixed?

28. Two men set out at the same time to walk, one from A to B, and the other from B to A, a distance of a miles. The former walks at the rate of p miles, and the latter at the rate of q miles an hour: at what distance from A will they meet?

29. A train runs from A to B in 3 hours; a second train runs from A to C, a point 15 miles beyond B, in $3\frac{1}{2}$ hours, travelling at a speed which is less by 1 mile per hour. Find distance from A to B.

30. Coffee is bought at 36 cents and chicory at 9 cents per lb.: in what proportion must they be mixed that 10 per cent may be gained by selling the mixture at 33 cents per lb.?

31. A man has one kind of coffee at a cents per pound, and another at b cents per pound. How much of each must he take to form a mixture of $a - b$ lbs., which he can sell at c cents a pound without loss?

32. A man spends c half-dollars in buying two kinds of silk at a dimes and b dimes a yard respectively; he could have bought 3 times as much of the first and half as much of the second for the same money. How many yards of each did he buy?

33. A man rides one-third of the distance from A to B at the rate of a miles an hour, and the remainder at the rate of $2b$ miles an hour. If he had travelled at a uniform rate of $3c$ miles an hour, he could have ridden from A to B and back again in the same time. Prove that

$$\frac{2}{c} = \frac{1}{a} + \frac{1}{b}.$$

34. A, B, C are three towns forming a triangle. A man has to walk from one to the next, ride thence to the next, and drive thence to his starting-point. He can walk, ride, and drive a mile in a, b, c minutes respectively. If he starts from B he takes $a + c - b$ hours, if he starts from C he takes $b + a - c$ hours, and if he starts from A he takes $c + b - a$ hours. Find the length of the circuit.

MISCELLANEOUS EXAMPLES IV.

1. Distinguish between *like* and *unlike* terms. Pick out the like terms in the expression $a^3 - 3ab + b^2 - 2a^3 + 3b^2 + 5ab + 7a^3$.

2. Subtract $-2a^3 + 3a^2b + 5b^3 - 4ab^2$ from $-1 - 2ab^2 + 3b^3$ and multiply the result by $-1 + 2a - b$.

3. Divide $8x^3 - 8x^2y + 4xy^2 - y^3$ by $2x - y$.

4. If the number of dollars I possess is represented by $+a$, what will $-a$ denote?

5. Factor the following expressions:

(i.) $a^2 - 64$,

(ii.) $a^3 - 27$.

6. Find the value of $\dfrac{1}{6x-2} - \dfrac{1}{2x-\frac{2}{3}} + \dfrac{1}{3x-1}$.

7. Solve $\dfrac{17-3x}{5} - \dfrac{2+4x}{3} = \dfrac{14+7x}{3} + 5 - 6x$.

8. There is a number of two digits which when divided by the unit digit gives a quotient 6; but if the digits be inverted the number is increased by 36: find the number.

9. Find the H. C. F. of $4x^3 - 16x^2 + 13x - 3$
and $3x^3 - 13x^2 + 13x - 3$.

10. Simplify $(\sqrt[3]{a^8}) \times (\sqrt[5]{a^7}) \times a^{-\frac{2}{3}} \div a^{\frac{3}{5}}$.

11. Find the value of $20\sqrt{\frac{1}{2}} + 14\sqrt{\frac{3}{7}} + 2\sqrt{21} - 7\sqrt{8} + \sqrt[3]{\frac{1}{2}} + \sqrt[3]{\frac{1}{16}}$.

12. Simplify $\dfrac{1}{x - \dfrac{2}{x+\frac{1}{2}}} \times \dfrac{1}{2+\dfrac{1}{x}} \div \dfrac{x}{2x - \dfrac{x+4}{x+1}}$.

13. Find the value of $\left[a^2 - \left\{(3b-c) + b^2 - \dfrac{2a}{b}\right\} - c^2\right]$ when $a = 2$, $b = 3$, $c = 4$.

14. Solve $\dfrac{a}{x-a} + \dfrac{b}{x+b} = \dfrac{c}{x-a}$.

15. Find the L. C. M. of $1-x$, $1-x^2$, $1-x^3$, and $(1-x)^3$.

16. Solve
$$\frac{3x}{5} + \frac{2y}{3} = 3\tfrac{2}{3},$$
$$\frac{2x}{3} - \frac{2y}{5} = 1\tfrac{2}{9}.$$

17. Simplify $\dfrac{3x}{2} - y - \left\{2x - \dfrac{y}{2} - 7 - \left(\dfrac{x}{2} - 4\right) + 2 - \dfrac{x}{2}\right\}$.

18. Find the square root of $x^6 + 8x^4 - 2x^3 + 16x^2 - 8x + 1$.

19. Solve the equations

(i.) $\dfrac{3x}{2} - \dfrac{5}{7} = 21x - \left(\dfrac{2x}{3} + \dfrac{143}{42}\right)$.

(ii.) $2\left(\dfrac{5x}{3} - 1\right) + \dfrac{11}{5} + \dfrac{14x}{15} = \dfrac{2x+7}{5} - 7$.

20. Expand the following binomials:

(i.) $(x+3a)^4$,

(ii.) $\left(2x - \dfrac{a}{2}\right)^5$.

21. The sum of the two digits of a number is 8 times their difference; if the digits be inverted, the number is diminished by 18: find the number.

22. Find the factors of (i.) $x^2 - 9x - 36$, (ii.) $2x^2 - 3x - 14$, and (iii.) $a^4b^4 - 7a^2b^2x^2 + x^4$.

23. Rationalize the denominator of $\dfrac{3 + 3\sqrt{5}}{5 - 2\sqrt{3}}$, and simplify $\sqrt{13 + 4\sqrt{3}}$.

24. Simplify $\dfrac{x^4 + 3x^3 - 11x^2 - 3x + 10}{x^3 + 3x^2 - 6x - 8}$.

For what values of x will both numerator and denominator vanish?

25. Solve the equations

$$\text{(i.)}\quad \begin{aligned} 2x + 3y + 4z &= 31, \\ x + 4y + z &= 18, \\ 3x + y + 2z &= 16. \end{aligned}$$

$$\text{(ii.)}\quad \frac{x-y}{2} + \frac{x+y}{3} = \frac{25}{6},\ x + y - 5 = \tfrac{2}{3}(y - x).$$

26. Simplify $\sqrt{-8} + \sqrt{-\frac{1}{2}} - \sqrt{-18} + \sqrt{-2 + 2\sqrt{-3}}$.

27. Simplify $\left(2x - \dfrac{x^2 - y^2}{x}\right)\left(3y + \dfrac{x^2 + y^2}{y}\right) \div \left(\dfrac{x^2}{y^2} + 5 + \dfrac{4y^2}{x^2}\right)$.

28. Find the middle term of the expansion of $\left(\dfrac{a}{x} + \dfrac{x}{a}\right)^{10}$.

29. Simplify $$\frac{\dfrac{a^3}{b^3} - \dfrac{b^3}{a^3}}{\left(\dfrac{a}{b} - \dfrac{b}{a}\right)\left(\dfrac{a}{b} + \dfrac{b}{a} - 1\right)} \times \frac{\dfrac{1}{b} - \dfrac{1}{a}}{\dfrac{1}{a^2} + \dfrac{1}{b^2} + \dfrac{1}{ab}}.$$

30. Solve the equations

(i.) $a(x - a) - b(x - b) = (a + b)(x - a - b)$.

(ii.) $(a + b)x - ay = a^2$, $(a^2 + b^2)x - aby = a^3$.

31. A sum of $10.10 is divided among 7 women and 10 men; the same sum could have been divided among 23 women and 4 men. Find how much each woman and man receives.

32. Find the cube root of $8x^6 + 12x^5 + 18x^4 + 13x^3 + 9x^2 + 3x + 1$, and the square root of $y^2 + 4y + 10 + \dfrac{12}{y} + \dfrac{9}{y^2}$.

33. Simplify $$\sqrt[a+b]{\left(\frac{x^a}{x^b}\right)^{ab} \div \left\{\frac{(x^{a-b})^a}{(x^{a+b})^b}\right\}^b}.$$

34. Simplify $\sqrt{59 - 24\sqrt{6}} + [(\sqrt{-6} + \sqrt{-3})(\sqrt{-3} + 2\sqrt{-1})]$.

CHAPTER XXVI.

Quadratic Equations.

281. Suppose the following problem were proposed for solution:

A dealer bought a number of horses for \$280. If he had bought four less, each would have cost \$8 more; how many did he buy?

We should proceed thus:

Let $x =$ the number of horses; then $\frac{280}{x} =$ the number of dollars each cost.

If he had bought 4 less, he would have had $x - 4$ horses, and each would have cost $\frac{280}{x-4}$ dollars.

$$\therefore\ 8 + \frac{280}{x} = \frac{280}{x-4};$$

whence $$x(x-4) + 35(x-4) = 35x;$$

$$\therefore\ x^2 - 4x + 35x - 140 = 35x;$$

$$\therefore\ x^2 - 4x = 140.$$

This equation involves the *square* of the unknown quantity; and in order to complete the solution of the problem we must discover a method of solving such equations.

282. Definition. An equation which contains the square of the unknown quantity, *but no higher power*, is called a **quadratic equation**, or an **equation of the second degree**.

If the equation contains both the square and the first power of the unknown, it is called an **affected quadratic**; if it contains only the square of the unknown it is said to be a **pure quadratic.**

Thus $2x^2 - 5x = 3$ is an affected quadratic,
and $5x^2 = 20$ is a pure quadratic.

PURE QUADRATIC EQUATIONS.

283. A *pure quadratic* may be considered as a simple equation in which the *square* of the unknown quantity is to be found.

Ex. Solve $$\frac{9}{x^2 - 27} = \frac{25}{x^2 - 11}.$$

Multiplying across, $9x^2 - 99 = 25x^2 - 675$;

transposing, $16x^2 = 576$;

$\therefore\ x^2 = 36$;

and taking the square root of these equals, we have

$$x = \pm 6.$$

NOTE. We prefix the double sign to the number on the right-hand side for the reason given in Art. 196.

284. In extracting the square root of the two sides of the equation $x^2 = 36$, it might seem that we ought to prefix the double sign to the quantities on both sides, and write $\pm x = \pm 6$. But an examination of the various cases shows this to be unnecessary. For $\pm x = \pm 6$ gives the four cases:

$$+x = +6,\ +x = -6,\ -x = +6,\ -x = -6,$$

and these are all included in the two already given, namely, $x = +6$, $x = -6$. Hence when we extract the square root of the two sides of an equation, it is sufficient to put the double sign before the square root of *one* side.

EXAMPLES XXVI. a.

Solve the following equations:

1. $4x^2 + 5 = x^2 + 17.$

2. $3x^2 + 3 = \dfrac{2x^2}{3} + 24.$

3. $(x + 1)(x - 1) = 2x^2 - 4.$

4. $\dfrac{2x^2-6}{2} - \dfrac{x^2-4}{4} - \dfrac{5x^2-10}{7} = 0.$

5. $x^2 + 2 = \dfrac{(x-1)^3 - x + 24}{x + 2}.$

6. $\dfrac{x-a}{x+a}+\dfrac{x+a}{x-a}=5.$

7. $\dfrac{3(x^2-1)}{x^2-1}+\dfrac{4(x^2-4)}{x^2+3}-\dfrac{3(9x^2-1)}{(x^2-1)(x^2+3)}=7.$

8. $(2x-c)(x+d)+(2x+c)(x-d)=2cd(2cd-1).$

AFFECTED QUADRATIC EQUATIONS.

285. The equation $x^2=36$ is an instance of the simplest form of quadratic equations. The equation $(x-3)^2=25$ may be solved in a similar way; for taking the square root of both sides, we have two *simple* equations,

$$x-3=\pm 5.$$

Taking the upper sign, $x-3=+5$, whence $x=8$;
taking the lower sign, $x-3=-5$, whence $x=-2$.
$\therefore$ the solution is $\qquad x=8$, or -2.

Now the given equation $(x-3)^2=25$
may be written $\qquad x^2-6x+(3)^2=25,$
or $\qquad x^2-6x=16.$

Hence, by retracing our steps, we learn that the equation

$$x^2-6x=16$$

can be solved by first adding $(3)^2$ to each side, and then extracting the square root; and we add 9 to each side because this quantity added to the left side makes it a *perfect square.*

Now whatever the quantity a may be,

$$x^2+2ax+a^2=(x+a)^2,$$

and

$$x^2-2ax+a^2=(x-a)^2;$$

so that, if a trinomial is a perfect square, and *its highest power,* x^2, *has unity for a coefficient,* the term without x must be equal to the *square of half the coefficient of* x.

Ex. 1. Solve $\qquad 7x=x^2-8.$

Transpose so as to have the terms involving x on one side, and the square term positive.

Thus $\qquad x^2-7x=8.$

Completing the square, $x^2 - 7x + (\frac{7}{2})^2 = 8 + \frac{49}{4}$;

that is, $$(x - \tfrac{7}{2})^2 = \tfrac{81}{4};$$
$$\therefore\ x - \tfrac{7}{2} = \pm\tfrac{9}{2};$$
$$\therefore\ x = \tfrac{7}{2} \pm \tfrac{9}{2} = 8, \text{ or } -1.$$

NOTE. We do not work out $(\frac{7}{2})^2$ on the left-hand side.

Ex. 2. Solve $32 - 3x^2 = 10x$.

Transposing, $3x^2 + 10x = 32$.

Divide throughout by 3, so as to make the coefficient of x^2 unity.

Thus $$x^2 + \tfrac{10}{3}x = \tfrac{32}{3};$$
completing the square, $$x^2 + \tfrac{10}{3}x + (\tfrac{5}{3})^2 = \tfrac{32}{3} + \tfrac{25}{9}\cdot$$
that is $$(x + \tfrac{5}{3})^2 = \tfrac{121}{9};$$
$$\therefore\ x + \tfrac{5}{3} = \pm\tfrac{11}{3};$$
$$\therefore\ x = -\tfrac{5}{3} \pm \tfrac{11}{3} = 2, \text{ or } -5\tfrac{1}{3}.$$

Ex. 3. Solve $7(x + 2a)^2 + 3a^2 = 5a(7x + 23a)$.

Simplifying, $7x^2 + 28ax + 28a^2 + 3a^2 = 35ax + 115a^2$,

that is, $$7x^2 - 7ax = 84a^2.$$
Whence $$x^2 - ax = 12a^2;$$
completing the square, $$x^2 - ax + \left(\frac{a}{2}\right)^2 = 12a^2 + \frac{a^2}{4};$$
that is, $$\left(x - \frac{a}{2}\right)^2 = \frac{49a^2}{4};$$
$$\therefore\ x - \frac{a}{2} = \pm\frac{7a}{2};$$
$$\therefore\ x = 4a, \text{ or } -3a.$$

286. We see then that the following are the steps required for solving an affected quadratic equation.

(1) *If necessary, simplify the equation so that the terms in* x^2 *and* x *are on one side of the equation, and the term without* x *on the other.*

(2) *Make the coefficient of* x^2 *unity and positive by dividing throughout by the coefficient of* x^2.

(3) *Add to each side of the equation the square of half the coefficient of* x.

(4) *Take the square root of each side.*

(5) *Solve the resulting simple equations.*

287. The quadratic equations considered hitherto have had two roots. Sometimes, however, there is only *one solu-*

tion. Thus if $x^2 - 2x + 1 = 0$, then $(x-1)^2 = 0$, whence $x = 1$ is the only solution. Nevertheless, in this and similar cases we find it convenient to say that the quadratic has *two equal roots.*

EXAMPLES XXVI. b.

1. $5x^2 + 14x = 55$.
2. $3x^2 + 121 = 44x$.
3. $25x = 6x^2 + 21$.
4. $8x^2 + x = 30$.
5. $3x^2 + 35 = 22x$.
6. $x + 22 - 6x^2 = 0$.
7. $15 = 17x + 4x^2$.
8. $21 + x = 2x^2$.
9. $9x^2 - 143 - 6x = 0$.
10. $12x^2 = 29x - 14$.
11. $20x^2 = 12 - x$.
12. $19x = 15 - 8x^2$.
13. $21x^2 + 22x + 5 = 0$.
14. $50x^2 - 15x = 27$.
15. $18x^2 - 27x - 26 = 0$.
16. $5x^2 = 8x + 21$.
17. $15x^2 - 2ax = a^2$.
18. $21x^2 = 2ax + 3a^2$.
19. $6x^2 = 11kx + 7k^2$.
20. $12x^2 + 23kx + 10k^2 = 0$.
21. $12x^2 - cx - 20c^2 = 0$.
22. $2(x-3) = 3(x+2)(x-3)$.
23. $(x+1)(2x+3) = 4x^2 - 22$.
24. $(3x-5)(2x-5) = x^2 + 2x - 3$.
25. $a^2x^2 - 2ax + a^2 = b$.
26. $cdx^2 = c^2x + d^2x - cd$.
27. $\frac{5x-1}{x+1} = \frac{3x}{2}$.
28. $\frac{3x-8}{x-2} = \frac{5x-2}{x+5}$.
29. $\frac{5x-7}{7x-5} = \frac{x-5}{2x-13}$.
30. $\frac{x+3}{2x-7} - \frac{2x-1}{x-3} = 0$.
31. $\frac{x+4}{x-4} + \frac{x-2}{x-3} = 6\frac{1}{3}$.
32. $\frac{1}{3-x} - \frac{4}{5} = \frac{1}{9-2x}$.
33. $\frac{x}{x+3} + \frac{2}{x+6} = \frac{13}{20}$.
34. $\frac{5}{x+1} - \frac{8}{x+2} = -\frac{x}{2x+4}$.
35. $\frac{a^2x^2}{b^2} - \frac{2ax}{c} + \frac{b^2}{c^2} = 0$.
36. $\frac{1}{c+x} + \frac{1}{d+x} = \frac{c+d}{cd}$.
37. $\frac{2x+5}{3x-2} - \frac{2x+7}{3x-4} + \frac{3}{4} = 0$.
38. $\frac{x+b}{x+c} + \frac{x+c}{x+b} = \frac{5}{2}$.
39. $\frac{2x}{x-1} + \frac{3x-1}{x+2} = \frac{5x-11}{x-2}$.
40. $\frac{x-3}{x+3} - \frac{x+3}{x-3} + 6\frac{6}{7} = 0$.
41. $\frac{3x+1}{x+8} + \frac{x-8}{3x-1} = \frac{17}{12}$.
42. $\frac{21x^3 - 16}{3x^2 - 4} - 7x = 5$.

288. Solution by Formula. After suitable reduction and transposition every quadratic equation can be written in the form

$$ax^2 + bx + c = 0,$$

where $a, b, c,$ may have *any* numerical values whatever. If therefore we can solve this quadratic, we can solve any.

Transposing, $ax^2 + bx = -c$; . . . . (1)

dividing by a, $x^2 + \frac{b}{a}x = -\frac{c}{a}$.

Completing the square by adding to each side $\left(\frac{b}{2a}\right)^2$,

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a};$$

that is, $$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2};$$

extracting the square root,

$$x + \frac{b}{2a} = \frac{\pm\sqrt{(b^2 - 4ac)}}{2a};$$

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}.$$

NOTE. The student will observe that b, the first term of the numerator of the fraction, is the coefficient of x in equation (1) *with its sign changed*, and that $4ac$, under the radical, is *plus* or *minus* according as the signs of a and c in equation (1) are *like* or *unlike*.

289. Instead of going through the process of completing the square in each particular example, we may now make use of this general formula, adapting it to the case in question by substituting the values of a, b, c.

Ex. Solve $5x^2 + 11x - 12 = 0$.

Here $a = 5$, $b = 11$, $c = -12$.

$$\therefore x = \frac{-11 \pm \sqrt{(11)^2 - 4.5(-12)}}{10}$$

$$= \frac{-11 \pm \sqrt{361}}{10} = \frac{-11 \pm 19}{10} = \frac{4}{5}, \text{ or } -3.$$

290. In the result $x = \dfrac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$,

it must be remembered that the expression $\sqrt{(b^2 - 4ac)}$ is the square root of the compound quantity $b^2 - 4ac$, *taken as a whole.* We cannot simplify the solution unless we know the numerical values of a, b, c. It may sometimes happen that these values do not make $b^2 - 4ac$ a perfect square. In such a case the exact numerical solution of the equation cannot be determined.

Ex. 1. Solve $$5x^2 - 15x + 11 = 0.$$

We have
$$x = \frac{15 \pm \sqrt{(-15)^2 - 4 \cdot 5 \cdot 11}}{2 \cdot 5}$$
$$= \frac{15 \pm \sqrt{5}}{10}.$$

Now $$\sqrt{5} = 2.236 \text{ approximately.}$$
$$\therefore\ x = \frac{15 \pm 2.236}{10} = 1.7236, \text{ or } 1.2764.$$

These solutions are correct only to four places of decimals, and neither of them will be found to *exactly* satisfy the equation.

Unless the *numerical* values of the unknown quantity are required it is usual to leave the roots in the form
$$\frac{15 + \sqrt{5}}{10},\ \frac{15 - \sqrt{5}}{10}.$$

Ex. 2. Solve $$x^2 - 3x + 5 = 0.$$

We have
$$x = \frac{3 \pm \sqrt{(-3)^2 - 4 \cdot 1 \cdot 5}}{2}$$
$$= \frac{3 \pm \sqrt{9 - 20}}{2} = \frac{3 \pm \sqrt{-11}}{2}.$$

But -11 has no square root exact or approximate [Art. 196]; so that no real value of x can be found to satisfy the equation. In such a case the roots are said to be *imaginary* or *impossible* [Art. 266].

291. Solution by Factoring. The following method will sometimes be found shorter than either of those already given.

Consider the equation $x^2 + \frac{7}{3}x = 2$.

Clearing of fractions, $3x^2 + 7x - 6 = 0$ . . . . (1);
by resolving the left-hand side into factors, we have
$$(3x - 2)(x + 3) = 0.$$

Now if *either* of the factors $3x - 2$, $x + 3$, be zero, their product is zero. Hence the quadratic equation is satisfied by either of the suppositions
$$3x - 2 = 0, \text{ or } x + 3 = 0.$$

Thus the roots are $\frac{2}{3},\ -3$.

From this we see that *when a quadratic equation has been simplified and brought to the form of equation* (1), its solution can be readily obtained if the expression on the left-hand side can be resolved into factors. Each of these factors equated to zero gives a simple equation, and a corresponding root of the quadratic.

Ex. 1. Solve $2x^2 - ax + 2bx = ab.$

Transposing, *so as to have all the terms on one side of the equation*, we have

$$2x^2 - ax + 2bx - ab = 0.$$

Now $2x^2 - ax + 2bx - ab = x(2x - a) + b(2x - a)$

$$= (2x - a)(x + b).$$

Therefore $(2x - a)(x + b) = 0;$

whence $2x - a = 0$, or $x + b = 0$.

$$\therefore\ x = \frac{a}{2}, \text{ or } -b.$$

Ex. 2. Solve $2(x^2 - 6) = 3(x - 4).$

We have $2x^2 - 12 = 3x - 12;$

that is, $2x^2 = 3x$ . . . . (1)

Transposing, $2x^2 - 3x = 0.$

$$x(2x - 3) = 0.$$

$$\therefore\ x = 0, \text{ or } 2x - 3 = 0.$$

Thus the roots are $0, \frac{3}{2}$.

NOTE. In equation (1) above we might have divided both sides by x and obtained the simple equation $2x = 3$, whence $x = \frac{3}{2}$, which is *one* of the solutions of the given equation. But the student must be particularly careful to notice that **whenever an x is removed by division from every term of an equation** *it must not be neglected, since the equation is satisfied by $x = 0$, which is therefore one of the roots.*

292. Formation of Equations with Given Roots.

It is now easy to form an equation whose roots are known.

Ex. Form the equation whose roots are 3 and $\frac{1}{2}$.

Here $x = 3$, or $x = \frac{1}{2};$

$$\therefore\ x - 3 = 0, \text{ or } x - \tfrac{1}{2} = 0;$$

both of these statements are included in

$$(x - 3)(x - \tfrac{1}{2}) = 0,$$

or $2x^2 - 7x + 3 = 0.$

From this it also appears that the factors of a trinomial, in the form $ax^2 + bx + c$, can be obtained by placing the expression equal to zero, solving the resulting quadratic equation (Art. 288), and subtracting each root separately from x. We shall return to the subject of this article in Chapter XXX.

293. Values found for the Unknown Quantity which do not satisfy the Original Equation.

From the following example it will be seen that in solving certain equations values may be obtained which will not satisfy the original equation.

Ex. Solve $\sqrt{x+5}+\sqrt{3x+4}=\sqrt{12x+1}$.

Squaring both sides,

$$x+5+3x+4+2\sqrt{(x+5)(3x+4)}=12x+1.$$

Transposing and dividing by 2,

$$\sqrt{(x+5)(3x+4)}=4x-4 \quad \ldots \ldots \quad (1).$$

Squaring, $(x+5)(3x+4)=16x^2-32x+16,$

or $13x^2-51x-4=0,$

$$(x-4)(13x+1)=0;$$

$$\therefore x=4, \text{ or } -\tfrac{1}{13}.$$

If we proceed to verify the solution by substituting these values in the original equation, it will be found that it is satisfied by $x=4$, but not by $x=-\frac{1}{13}$. But this latter value will be found on trial to satisfy the given equation if we alter the sign of the second radical; thus,

$$\sqrt{x+5}-\sqrt{3x+4}=\sqrt{12x+1}.$$

On squaring this and reducing, we obtain

$$-\sqrt{(x+5)(3x+4)}=4x-4 \quad \ldots \ldots \quad (2);$$

and a comparison of (1) and (2) shows that in the next stage of the work *the same quadratic equation is obtained* in each case, the roots of which are 4 and $-\frac{1}{13}$, as already found.

From this it appears that when the solution of an equation requires that both sides should be squared, we cannot be certain without trial which of the values found for the unknown quantity will satisfy the original equation.

In order that all the values found by the solution of the equation may be applicable, it will be necessary to take into account both signs of the radicals in the given equation.

EXAMPLES XXVI. c.

Solve by the aid of the formula in Art. 288:

1. $3x^2=15-4x$.

2. $2x^2+7x=15$.

3. $2x^2+7-9x=0$.

4. $x^2=3x+5$.

5. $5x^2+4+21x=0$.

6. $x^2+11=7x$.

7. $8x^2=x+7$.

8. $5x^2=17x-10$.

9. $35+9x-2x^2=0$.

10. $3x^2=x+1$.

11. $3x^2+5x=2$.

12. $2x^2+5x-33=0$.

Solve by resolution into factors:

13. $6x^2 = 7 + x.$
14. $21 + 8x^2 = 26x.$
15. $26x - 21 + 11x^2 = 0.$
16. $5x^2 + 26x + 24 = 0.$
17. $4x^2 = \frac{4}{15}x + 3.$
18. $x^2 - 2 = \frac{23}{12}x.$
19. $7x^2 = 28 - 96x.$
20. $96x^2 = 4x + 15.$
21. $25x^2 = 5x + 6.$
22. $35 - 4x = 4x^2.$
23. $12x^2 - 11ax = 36a^2.$
24. $12x^2 + 36a^2 = 43ax.$
25. $35b^2 = 9x^2 + 6bx.$
26. $36x^2 - 35b^2 = 12bx.$
27. $x^2 - 2ax + 4ab = 2bx.$
28. $x^2 - 2ax + 8x = 16a.$
29. $3x^2 - 2ax - bx = 0.$
30. $ax^2 + 2x = bx.$

Solve:

31. $\dfrac{23}{x+4} + \dfrac{3x}{11} = \dfrac{1}{3}(x+5).$

32. $\sqrt{3x+10} + \sqrt{x+2} = \sqrt{10x+16}.$

33. $\dfrac{3x-4}{x+1} - \dfrac{x-1}{3x+4} + \dfrac{1}{2} = 0.$

34. $bx^2 - \dfrac{6d^2}{b+c} = dx - cx^2.$

35. $x^2 + 2cx - 2dx = 2cd - d^2.$

36. $\sqrt{2x+6} - \sqrt{x+4} = \sqrt{x-4}.$

37. $\dfrac{1}{a+x} + \dfrac{1}{b+x} = \dfrac{a+b}{ab}.$

38. $2cx^2 + 2d^2(x+c) = dx(x+5c).$

39. $\dfrac{x+m}{x-m} + \dfrac{x-m}{x+m} = \dfrac{x^2+m^2}{x^2-m^2} + \dfrac{x^2-m^2}{x^2+m^2}.$

40. $\dfrac{1}{x+a+b} = \dfrac{1}{b} + \dfrac{1}{a} + \dfrac{1}{x}.$

41. $\sqrt{x+\frac{3}{4}} + \sqrt{3x+\frac{1}{4}} = \sqrt{6x+\frac{5}{2}}.$

42. $\sqrt{x+3} + \sqrt{2x+1} = 2\sqrt{3x-1}.$

43. $\dfrac{x-2}{x-3} + \dfrac{3x-11}{x-4} = \dfrac{4x+13}{x+1}.$

44. $\dfrac{k-n}{2m+x} + \dfrac{m-n}{2k+x} = \dfrac{k+m-2n}{k+m+x}.$

45. $(a-b)x^2 + (b-c)x + c - a = 0.$

46. $a(b-c)x^2 + b(c-a)x + c(a-b) = 0.$

47. $\sqrt{a-x} + \sqrt{b-x} = \sqrt{a+b-2x}.$

48. $\dfrac{1}{a-x} + \dfrac{1}{b-x} = \dfrac{1}{a-c} + \dfrac{1}{b-c}.$

49. $\sqrt{x-p} + \sqrt{x-q} = \dfrac{p}{\sqrt{x-q}} + \dfrac{q}{\sqrt{x-p}}.$

50. $\sqrt{(x-2)(x-3)} + 5\sqrt{\dfrac{x-2}{x-3}} = \sqrt{x^2+6x+8}.$

CHAPTER XXVII.

EQUATIONS IN QUADRATIC FORM.

294. An equation in the form $ax^{2n} + bx^n = c$, n being a positive or negative integer or fraction, is in **quadratic form.** Thus $x^4 + 4x^2 = 117$, $x^{\frac{4}{3}} + 7x^{\frac{2}{3}} = 44$, and $x^{-\frac{1}{2}} + x^{-\frac{1}{4}} = a$ are equations in quadratic form.

We give a few examples showing that the ordinary rules for quadratic equations are applicable to those in quadratic form.

Ex. **1.** Solve $x^4 - 13x^2 = -36$.

By formula [Art. 288] $x^2 = \dfrac{13 \pm \sqrt{(13)^2 - 4(36)}}{2}$

$$= \frac{13 \pm \sqrt{169 - 144}}{2}$$

$$= \frac{13 \pm 5}{2} = 9 \text{ or } 4;$$

$$\therefore\ x = \pm 3 \text{ or } \pm 2.$$

Ex. **2.** Solve $2x^{\frac{2}{3}} - 3x^{\frac{1}{3}} = 2$.

By formula $\qquad x^{\frac{1}{3}} = \dfrac{3 \pm \sqrt{9 + 16}}{4}$

$$= \frac{3 \pm 5}{4} = 2 \text{ or } -\frac{1}{2}.$$

Raising to the third power, $x = 8$, or $-\frac{1}{8}$.

Ex. **3.** Solve $2x^{-\frac{1}{2}} - 9x^{-\frac{1}{4}} = -4$.

By formula, $\qquad x^{-\frac{1}{4}} = \dfrac{9 \pm \sqrt{81 - 32}}{4} = \dfrac{9 \pm 7}{4} = 4 \text{ or } \frac{1}{2}.$

Raising to the fourth power,

$$x^{-1} = 256 \text{ or } \tfrac{1}{16};$$

that is, $\qquad \dfrac{1}{x} = 256 \text{ or } \tfrac{1}{16};$

$$\therefore\ x = \tfrac{1}{256} \text{ or } 16.$$

EXAMPLES XXVII. a.

1. $x^4 - 13x^2 + 36 = 0.$
2. $x^6 + 7x^3 = 8.$
3. $x^6 - 19x^3 = 216.$
4. $8x^6 + 65x^3 + 8 = 0.$
5. $3\sqrt{x} - 3x^{-\frac{1}{2}} = 8.$
6. $27x^{\frac{3}{2}} - 1 = 26x^{\frac{3}{4}}.$
7. $x^4 - 74x^2 = -1225.$
8. $x^{-2} - 2x^{-1} = 8.$
9. $9 + x^{-4} = 10x^{-2}.$
10. $2\sqrt{x} + 2x^{-\frac{1}{2}} = 5.$
11. $6x^{\frac{3}{4}} = 7x^{\frac{1}{4}} - 2x^{-\frac{1}{4}}$
12. $x^{\frac{2}{n}} + 6 = 5x^{\frac{1}{n}}.$
13. $3x^{\frac{1}{2n}} - x^{\frac{1}{n}} - 2 = 0.$
14. $6\sqrt{x} = 5x^{-\frac{1}{2}} - 13.$
15. $1 + 8x^{\frac{6}{5}} + 9\sqrt[5]{x^3} = 0.$
16. $8x^{\frac{3}{2n}} - 8x^{-\frac{3}{2n}} = 63.$

295. Any equation which can be thrown into the form

$$ax^2 + bx + c + p\sqrt{ax^2 + bx + c} = q$$

may be solved as follows. Putting $y = \sqrt{ax^2 + bx + c}$, we obtain

$$y^2 + py - q = 0.$$

Let r_1 and r_2 be the roots of this equation, so that

$$\sqrt{ax^2 + bx + c} = r_1, \quad \sqrt{ax^2 + bx + c} = r_2;$$

from these equations we shall obtain *four* values of x.

When no sign is prefixed to a radical, it is usually understood that it is to be taken as positive; hence, if r_1 and r_2 are both positive, all the four values of x satisfy the *original* equation. If, however, r_1 or r_2 is negative, the roots found from the resulting quadratic will satisfy the equation

$$ax^2 + bx + c - p\sqrt{ax^2 + bx + c} = q,$$

but not the original equation.

Ex. 1. Solve $x^2 - 5x + 2\sqrt{x^2 - 5x + 3} = 12.$

Add 3 to each side; then

$$x^2 - 5x + 3 + 2\sqrt{x^2 - 5x + 3} = 15.$$

Putting $\sqrt{x^2 - 5x + 3} = y$, we obtain $y^2 + 2y - 15 = 0$; whence $y = 3$ or -5.

Thus $\sqrt{x^2 - 5x + 3} = +3$, or $\sqrt{x^2 - 5x + 3} = -5.$

Squaring and solving the resulting quadratics, we obtain from the first $x = 6$ or -1; and from the second $x = \frac{5 \pm \sqrt{113}}{2}$. The first pair of values satisfies the given equation, but the second pair satisfies the equation

$$x^2 - 5x - 2\sqrt{x^2 - 5x + 3} = 12.$$

Ex. 2. Solve $3x^2 - 7 + 3\sqrt{3x^2 - 16x + 21} = 16x$.

Transposing, $3x^2 - 16x - 7 + 3\sqrt{3x^2 - 16x + 21} = 0$.

Add 28 to each side; then

$$3x^2 - 16x + 21 + 3\sqrt{3x^2 - 16x + 21} = 28.$$

Proceeding as in Ex. 1, we have

$$y^2 + 3y = 28; \text{ whence } y = 4 \text{ or } -7.$$

Thus $\sqrt{3x^2 - 16x + 21} = 4$ or $\sqrt{3x^2 - 16x + 21} = -7$.

Squaring and solving, we obtain

$$x = 5, \ \tfrac{1}{3}, \text{ or } \frac{8 \pm 2\sqrt{37}}{3}.$$

The values 5 and $\frac{1}{3}$ satisfy the original equation. The other values satisfy the equation

$$3x^2 - 7 - 3\sqrt{3x^2 - 16x + 21} = 16x.$$

296. Occasionally equations of the fourth degree may be arranged in expressions that will be in quadratic form.

Ex. Solve $x^4 - 8x^3 + 10x^2 + 24x + 5 = 0$.

This may be written $x^4 - 8x^3 + 16x^2 - 6x^2 + 24x = -5$,

or $$(x^2 - 4x)^2 - 6(x^2 - 4x) = -5;$$

by formula, $$x^2 - 4x = \frac{6 \pm \sqrt{36 - 20}}{2} = \frac{6 \pm 4}{2} = 5 \text{ or } 1;$$

whence $$x = 5, \ -1, \text{ or } 2 \pm \sqrt{5}.$$

The student will notice that in such examples he should divide the term containing x^3 by twice the square root of the first term and then square the result for the third term. In this case a third term of $16x^2$ is required, therefore we write the term $10x^2$ of the original equation in the form $16x^2 - 6x^2$.

297. Equations like the following are of frequent occurrence.

Ex. Solve
$$\frac{x^2-6}{x}+\frac{5x}{x^2-6}=6.$$

Write y for $\frac{x^2-6}{x}$; thus

$$y+\frac{5}{y}=6, \text{ or } y^2-6y+5=0;$$

whence
$$y=5, \text{ or } 1.$$

$$\therefore \frac{x^2-6}{x}=5, \text{ or } \frac{x^2-6}{x}=1;$$

that is,
$$x^2-5x-6=0, \text{ or } x^2-x-6=0.$$

Thus
$$x=6,\ -1; \text{ or } x=3,\ -2.$$

EXAMPLES XXVII. b.

Solve the following equations:

1. $x^2+x+1=\frac{42}{x^2+x}$.
2. $\frac{x}{x^2-1}+\frac{x^2-1}{x}=2\frac{1}{6}$.
3. $\left(x+\frac{1}{x}\right)^2-4\left(x+\frac{1}{x}\right)=5$.
4. $\frac{x^2-3}{x}+\frac{3x}{x^2-3}=\frac{13}{2}$.
5. $x^2+2\sqrt{x^2+6x}=24-6x$.
6. $\left(x-\frac{6}{x}\right)^2+4x-\frac{24}{x}=5$.
7. $27x^{\frac{3}{2}}-4=26x^{\frac{3}{4}}$.
8. $x^2+3x-\frac{20}{x^2+3x}=8$.
9. $3x^2-4x+\sqrt{3x^2-4x-6}=18$.
10. $2x^2-2x+2\sqrt{2x^2-7x+6}=5x-6$.
11. $x^2+6\sqrt{x^2-2x+5}=11+2x$.
12. $2\sqrt{x^2-6x+2}+4x+1=x^2-2x$.
13. $\sqrt{4x^2+2x+7}=12x^2+6x-119$.
14. $3x(3-x)=11-4\sqrt{x^2-3x+5}$.
15. $x^2-x+3\sqrt{2x^2-3x+2}=\frac{x}{2}+7$.
16. $2x^2-2x-17+2\sqrt{2x^2-3x+7}=x$.
17. $2x^2+3x+1=\frac{30}{2x^2+3x}$.
18. $x^4-8x^3-12x^2+112x=128$.
19. $x^4+2x^3-3x^2-4x-96=0$.
20. $x^4-10x^3+30x^2-25x+4=0$.
21. $x^4-14x^3+61x^2-84x+20=0$.

CHAPTER XXVIII.

SIMULTANEOUS EQUATIONS, INVOLVING QUADRATICS.

298. We shall now consider some of the most useful methods of solving simultaneous equations, one or more of which may be of a degree higher than the first; but no fixed rules can be laid down which are applicable to all cases.

299. Equations solved by finding the Values of $(x+y)$ and $(x-y)$.

Ex. 1. Solve
$$x+y=15 \quad \ldots\ldots\ldots \quad (1),$$
$$xy=36 \quad \ldots\ldots\ldots \quad (2).$$

From (1) by squaring, $x^2+2xy+y^2=225$;
from (2), $4xy=144$;
by subtraction, $x^2-2xy+y^2=81$;
by taking the square root, $x-y=\pm 9$.

Combining this with (1) we have to consider the two cases,

$$\left.\begin{aligned} x+y&=15,\\ x-y&=9.\end{aligned}\right\} \qquad \left.\begin{aligned} x+y&=15,\\ x-y&=-9.\end{aligned}\right\}$$

from which we find
$$\left.\begin{aligned} x&=12,\\ y&=3.\end{aligned}\right\} \qquad \left.\begin{aligned} x&=3,\\ y&=12.\end{aligned}\right\}$$

Ex. 2. Solve
$$x-y=12 \quad \ldots\ldots\ldots \quad (1),$$
$$xy=85 \quad \ldots\ldots\ldots \quad (2).$$

From (1), $x^2-2xy+y^2=144$;
from (2), $4xy=340$;
by addition, $x^2+2xy+y^2=484$;
by taking the square root, $x+y=\pm 22$.

Combining this with (1) we have the two cases,

$$\left.\begin{aligned} x+y&=22,\\ x-y&=12.\end{aligned}\right\} \qquad \left.\begin{aligned} x+y&=-22,\\ x-y&=12.\end{aligned}\right\}$$

Whence
$$\left.\begin{aligned} x&=17,\\ y&=5.\end{aligned}\right\} \qquad \left.\begin{aligned} x&=-5,\\ y&=-17.\end{aligned}\right\}$$

300. These are the simplest cases that arise, but they are specially important since the solution in a large number of other cases is dependent upon them.

As a rule our object is to solve the proposed equations *symmetrically*, by finding the values of $x+y$ and $x-y$. From the foregoing examples it will be seen that we can always do this as soon as we have obtained the product of the unknowns, and either their sum or their difference.

Ex. 1. Solve
$$x^2+y^2=74 \quad \ldots\ldots \quad (1),$$
$$xy=35 \quad \ldots\ldots \quad (2).$$

Multiply (2) by 2, then by addition and subtraction we have
$$x^2+2xy+y^2=144.$$
$$x^2-2xy+y^2=4.$$

Whence
$$x+y=\pm 12,$$
$$x-y=\pm 2.$$

We have now four cases to consider; namely,
$$\left.\begin{array}{l}x+y=12,\\ x-y=2.\end{array}\right\} \quad \left.\begin{array}{l}x+y=12,\\ x-y=-2.\end{array}\right\} \quad \left.\begin{array}{l}x+y=-12,\\ x-y=2.\end{array}\right\} \quad \left.\begin{array}{l}x+y=-12,\\ x-y=-2.\end{array}\right\}$$

From which the values of x are $7,\ 5,\ -5,\ -7$;
and the corresponding values of y are $5,\ 7,\ -7,\ -5$.

Ex. 2. Solve
$$x^2+y^2=185 \quad \ldots\ldots \quad (1),$$
$$x+y=17 \quad \ldots\ldots \quad (2).$$

By subtracting (1) from the square of (2) we have
$$2xy=104;$$
$$\therefore\ xy=52 \quad \ldots\ldots \quad (3).$$

Equations (2) and (3) can now be solved by the method of Art. 299, Ex. 1; and the solution is
$$\left.\begin{array}{l}x=13, \text{ or } 4,\\ y=4, \text{ or } 13.\end{array}\right\}$$

EXAMPLES XXVIII. a.

Solve the following equations:

1. $x+y=28$, $xy=187$.
2. $x+y=51$, $xy=518$.
3. $x+y=74$, $xy=1113$.
4. $x-y=5$, $xy=126$.
5. $x-y=8$, $xy=513$.
6. $xy=1075$, $x-y=18$.

7. $xy = 923,$ $x + y = 84.$

8. $x - y = -8,$ $xy = 1353.$

9. $x - y = -22,$ $xy = 3848.$

10. $xy = -2193,$ $x + y = -8.$

11. $x - y = -18,$ $xy = 1363.$

12. $xy = -1914,$ $x + y = -65.$

13. $x^2 + y^2 = 89,$ $xy = 40.$

14. $x^2 + y^2 = 170,$ $xy = 13.$

15. $x^2 + y^2 = 65,$ $xy = 28.$

16. $x^2 + y^2 = 178,$ $x + y = 16.$

17. $x + y = 15,$ $x^2 + y^2 = 125.$

18. $x - y = 4,$ $x^2 + y^2 = 106.$

19. $x^2 + y^2 = 180,$ $x - y = 6.$

20. $x^2 + y^2 = 185,$ $x - y = 3.$

21. $x + y = 13,$ $x^2 + y^2 = 97.$

22. $x + y = 9,$ $x^2 + xy + y^2 = 61.$

23. $x - y = 3,$ $x^2 - 3xy + y^2 = -19.$

24. $x^2 - xy + y^2 = 76,$ $x + y = 14.$

25. $\frac{1}{10}(x - y) = 1,$ $x^2 - 4xy + y^2 = 52.$

26. $\frac{1}{x} + \frac{1}{y} = 2,$ $x + y = 2.$

27. $\frac{1}{x} + \frac{1}{y} = \frac{7}{12},$ $xy = 12.$

28. $ax + by = 2,$ $abxy = 1.$

29. $x^2 + pxy + y^2 = p + 2,$ $qx^2 + xy + qy^2 = 2q + 1.$

301. Equations which can be reduced to One of the Cases already considered. Any pair of equations of the form

$$x^2 \pm pxy + y^2 = a^2 \quad \text{. . . . .} \quad (1),$$

$$x \pm y = b \quad \text{. . . . .} \quad (2),$$

where p is any numerical quantity, can be reduced to one of the cases already considered; for, by squaring (2) and combining with (1), an equation to find xy is obtained; the solution can then be completed by the aid of equation (2).

Ex. 1. Solve $\quad x^3 - y^3 = 999 \quad \text{. . . . .} \quad (1),$

$$x - y = 3 \quad \text{. . . . .} \quad (2).$$

By division, $\quad x^2 + xy + y^2 = 333 \quad \text{. . . . . . .} \quad (3);$

from (2), $\quad x^2 - 2xy + y^2 = 9;$

by subtraction, $\quad 3xy = 324,$

$$xy = 108 \quad \text{. . . . .} \quad (4).$$

From (2) and (4), $\quad \left.\begin{array}{l} x = 12, \text{ or } -9, \\ y = 9, \text{ or } -12. \end{array}\right\}$

Ex. 2. Solve
$$x^4 + x^2y^2 + y^4 = 2613 \quad \ldots\ldots \quad (1),$$
$$x^2 + xy + y^2 = 67 \quad \ldots\ldots \quad (2).$$

Dividing (1) by (2), $x^2 - xy + y^2 = 39 \quad \ldots\ldots \quad (3).$

From (2) and (3), by addition, $x^2 + y^2 = 53$;

by subtraction, $xy = 14$;

whence
$$\left.\begin{array}{l} x = \pm 7,\ \pm 2, \\ y = \pm 2,\ \pm 7. \end{array}\right\} \quad \text{[Art. 300, Ex. 1.]}$$

Ex. 3. Solve
$$\frac{1}{x} - \frac{1}{y} = \frac{1}{3} \quad \ldots\ldots \quad (1),$$
$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{9} \quad \ldots\ldots \quad (2).$$

From (1), by squaring, $\dfrac{1}{x^2} - \dfrac{2}{xy} + \dfrac{1}{y^2} = \dfrac{1}{9}$;

by subtraction, $\dfrac{2}{xy} = \dfrac{4}{9}$;

adding to (2), $\dfrac{1}{x^2} + \dfrac{2}{xy} + \dfrac{1}{y^2} = 1$;

$$\therefore \frac{1}{x} + \frac{1}{y} = \pm 1.$$

Combining with (1),
$$\frac{1}{x} = \frac{2}{3}, \text{ or } -\frac{1}{3},$$
$$\frac{1}{y} = \frac{1}{3}, \text{ or } -\frac{2}{3};$$
$$\therefore \left.\begin{array}{l} x = \frac{3}{2}, \text{ or } -3, \\ y = 3, \text{ or } -\frac{3}{2}. \end{array}\right\}$$

EXAMPLES XXVIII. b.

1. $x^3 + y^3 = 407,$ $x + y = 11.$

2. $x^3 + y^3 = 637,$ $x + y = 13.$

3. $x + y = 23,$ $x^3 + y^3 = 3473.$

4. $x^3 - y^3 = 218,$ $x - y = 2.$

5. $x - y = 4,$ $x^3 - y^3 = 988.$

6. $x^3 - y^3 = 2197,$ $x - y = 13.$

7. $x^4 + x^2y^2 + y^4 = 2128,$ $x^2 + xy + y^2 = 76.$

8. $x^4 + x^2y^2 + y^4 = 2923,$ $x^2 - xy + y^2 = 37.$

9. $x^4 + x^2y^2 + y^4 = 9211,$ $x^2 - xy + y^2 = 61.$

10. $x^4 + x^2y^2 + y^4 = 7371,$ $x^2 - xy + y^2 = 63.$

11. $\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{481}{576}$,
$\dfrac{1}{x}+\dfrac{1}{y}=\dfrac{29}{24}$.

12. $\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{61}{900}$,
$xy=30$.

13. $\dfrac{x}{y}+\dfrac{y}{x}=2\frac{1}{2}$,
$x+y=6$.

14. $\dfrac{x}{y}+\dfrac{y}{x}=2\frac{16}{21}$,
$x-y=4$.

15. $\dfrac{34}{x^2+y^2}=\dfrac{15}{xy}$,
$x+y=8$.

16. $x^3-y^3=56$,
$x^2+xy+y^2=28$.

17. $4(x^2+y^2)=17xy$,
$x-y=6$.

18. $x^3+y^3=126$,
$x^2-xy+y^2=21$.

19. $\dfrac{1}{x^3}+\dfrac{1}{y^3}=1\frac{1}{125}$,
$\dfrac{1}{x}+\dfrac{1}{y}=1\frac{1}{5}$.

20. $\dfrac{1}{x^3}-\dfrac{1}{y^3}=91$,
$\dfrac{1}{x}-\dfrac{1}{y}=1$.

302. Homogeneous Equations of the Same Degree. The following method of solution may always be used when the equations are *of the same degree and homogeneous.*

Ex. Solve $$x^2+xy+2y^2=74 \quad \ldots \quad (1),$$
$$2x^2+2xy+y^2=73 \quad \ldots \quad (2).$$

Put $y=mx$, and substitute in both equations. Thus
$$x^2(1+m+2m^2)=74 \quad \ldots \quad (3),$$
and $$x^2(2+2m+m^2)=73 \quad \ldots \quad (4).$$

By division, $$\frac{1+m+2m^2}{2+2m+m^2}=\frac{74}{73};$$
$$\therefore\ 73+73m+146m^2=148+148m+74m^2;$$
$$\therefore\ 72m^2-75m-75=0,$$
or $$24m^2-25m-25=0;$$
$$\therefore\ (8m+5)(3m-5)=0;$$
$$\therefore\ m=-\tfrac{5}{8}, \text{ or } \tfrac{5}{3}.$$

(i.) Take $m=-\frac{5}{8}$, and substitute in either (3) or (4).

From (3) $$x^2(1-\tfrac{5}{8}+\tfrac{50}{64})=74;$$
$$\therefore\ x^2=\frac{64\times 74}{74}=64;$$
$$\therefore\ x=\pm 8;$$
$$\therefore\ y=mx=-\tfrac{5}{8}\times\pm 8=\mp 5.$$

(ii.) Take $m=\frac{5}{3}$; then from (3), $x^2(1+\frac{5}{3}+\frac{50}{9})=74$,
$$x^2=\frac{74\times 9}{74}=9;$$
$$\therefore\ x=\pm 3;\ \therefore\ y=mx=\tfrac{5}{3}\times\pm 3=\pm 5.$$

The student will notice that, having found the values of x, we obtained those of y from the equation $y = mx$, *using, in each case, the value of m employed in finding those particular values of x.*

303. Equations of which One is of the First Degree and the Other of a Higher Degree. We may from the simple equation find the value of one of the unknowns in terms of the other, and substitute in the second equation.

Ex. Solve $\quad 3x - 4y = 5 \quad . \quad . \quad . \quad (1),$

$$3x^2 - xy - 3y^2 = 21 \quad . \quad . \quad . \quad (2).$$

From (1) we have $\quad x = \frac{5 + 4y}{3};$

and substituting in (2), $\quad \frac{3(5 + 4y)^2}{9} - \frac{y(5 + 4y)}{3} - 3y^2 = 21;$

$$\therefore 75 + 120y + 48y^2 - 15y - 12y^2 - 27y^2 = 189;$$

$$9y^2 + 105y - 114 = 0;$$

$$3y^2 + 35y - 38 = 0;$$

$$\therefore (y - 1)(3y + 38) = 0;$$

$$\therefore y = 1, \text{ or } -\tfrac{38}{3};$$

and by substituting in (1), $\quad x = 3, \text{ or } -\tfrac{137}{9}.$

304. Symmetrical Equations. The following method of solution may *always* be used when the given equations are *symmetrical,* that is, when the unknown quantities in each equation may be interchanged without destroying the equality. The same method may generally be employed with advantage where the given equations are symmetrical except with respect to the *signs* of the terms.

Ex. Solve $\quad x^4 + y^4 = 82 \quad . \quad . \quad . \quad (1),$

$$x - y = 2 \quad . \quad . \quad . \quad (2).$$

Put $\quad x = u + v, \text{ and } y = u - v;$

then from (2) we obtain $\quad v = 1.$

Substituting in (1), $\quad (u + 1)^4 + (u - 1)^4 = 82;$

$$\therefore 2(u^4 + 6u^2 + 1) = 82;$$

$$u^4 + 6u^2 - 40 = 0;$$

whence $u^2 = 4$, or -10;

and $u = \pm 2$, or $\pm\sqrt{-10}$.

Thus, $x = u + v = 3,\ -1,\ 1 \pm \sqrt{-10}$;

$y = u - v = 1,\ -3,\ -1 \pm \sqrt{-10}$.

NOTE. We may assume $x + y = 2u$ and $x - y = 2v$, u and v being *any* unknown quantities, whence we obtain $x = u + v$, and $y = u - v$, the values used in the above.

305. Miscellaneous Cases. The examples we have given will be sufficient as a general explanation of the methods to be employed; but in some cases special artifices are necessary.

Ex. 1. Solve $x^2y^2 - 6x = 34 - 3y$ . . . . . . (1),

$3xy + y = 2(9 + x)$ . . . . . . (2).

From (1), $x^2y^2 - 6x + 3y = 34$;

from (2), $9xy - 6x + 3y = 54$;

by subtraction, $x^2y^2 - 9xy = -20$,

$$xy = \frac{9 \pm \sqrt{81 - 80}}{2} = \frac{9 \pm 1}{2} = 5 \text{ or } 4.$$

(i.) Substituting $xy = 5$ in (2) gives $y - 2x = 3$.

From these equations we obtain $\left.\begin{aligned} x &= 1, \text{ or } -\tfrac{5}{2}, \\ y &= 5, \text{ or } -2. \end{aligned}\right\}$

(ii.) Substituting $xy = 4$ in (2) gives $y - 2x = 6$.

From these equations we obtain $\left.\begin{aligned} x &= \frac{-3 \pm \sqrt{17}}{2}, \\ \text{and} \quad y &= 3 \pm \sqrt{17}. \end{aligned}\right\}$

Ex. 2. Solve $y^2 + yz + z^2 = 49$ . . . . (1),

$z^2 + zx + x^2 = 19$ . . . . (2),

$x^2 + xy + y^2 = 39$ . . . . (3).

Subtracting (2) from (1),

$$y^2 - x^2 + z(y - x) = 30;$$

that is, $(y - x)(x + y + z) = 30$ . . . . (4).

Similarly from (1) and (3),

$(z - x)(x + y + z) = 10$ . . . . (5).

Hence from (4) and (5), by division,

$$\frac{y-x}{z-x}=3;$$

whence $$y=3z-2x.$$

Substituting in equation (3), we obtain

$$x^2-3xz+3z^2=13.$$

From (2), $$x^2+xz+z^2=19.$$

Solving these homogeneous equations, we obtain

$$x=\pm 2,\ z=\pm 3;\ \text{and therefore } y=\pm 5;$$

or $$x=\pm\frac{11}{\sqrt{7}},\ z=\pm\frac{1}{\sqrt{7}};\ \text{and therefore } y=\mp\frac{19}{\sqrt{7}}.$$

EXAMPLES XXVIII. c.

1. $5x-y=17,\quad xy=12.$

2. $x^2+xy=15,\quad y^2+xy=10.$

3. $x-y=10,\quad x^2-2xy-3y^2=84.$

4. $3x+2y=16,\quad xy=10.$

5. $3x-y=11,\quad 3x^2-y^2=47.$

6. $x-3y=1,\quad x^2-2xy+9y^2=17.$

7. $x+2y=9,\quad 3y^2-5x^2=43.$

8. $x^2+y^2=5,\quad 2xy-y^2=3.$

9. $5x+y=3,\quad 2x^2-3xy-y^2=1.$

10. $3x^2-5y^2=28,\quad 3xy-4y^2=8.$

11. $3x^2-y^2=23,\quad 2x^2-xy=12.$

12. $x^2+xy+y^2=3\frac{1}{4},\quad 2x^2-3xy+2y^2=2\frac{3}{4}.$

13. $x^2-3xy+y^2+1=0,\quad 3x^2-xy+3y^2=13.$

14. $7xy-8x^2=10,\quad 8y^2-9xy=18.$

15. $x^2-2xy=21,\quad xy+y^2=18.$

16. $x^2+3xy=54,\quad xy+4y^2=115.$

17. $x^3+y^3=152,\quad x^2y+xy^2=120.$

18. $x^3-y^3=127,\quad x^2y-xy^2=42.$

19. $x^3-y^3=208,\quad xy(x-y)=48.$

20. $x^2y^2+5xy=84,\quad x+y=8.$

21. $x^2+4y^2+80=15x+30y,\quad xy=6.$

22. $9x^2+y^2-63x-21y+128=0,\quad xy=4.$

23. $\dfrac{1}{x^2}+\dfrac{1}{y^2}=\dfrac{45}{4},\quad \dfrac{1}{x}-\dfrac{1}{y}=\dfrac{3}{2}.$

24. $\dfrac{1}{x^3}+\dfrac{1}{y^3}=\dfrac{243}{8},\quad \dfrac{1}{x}+\dfrac{1}{y}=\dfrac{9}{2}.$

25. $x^4+x^2y^2+y^4=931,\quad x^2-xy+y^2=19.$

26. $x^2+xy+y^2=84,\quad x-\sqrt{xy}+y=6.$

27. $x+\sqrt{xy}+y=65,\quad x^2+xy+y^2=2275.$

28. $x+y=7+\sqrt{xy},\quad x^2+y^2=133-xy.$

29. $3x^2-5y^2=7,\quad 3xy-4y^2=2.$

30. $5y^2-7x^2=17,\quad 5xy-6x^2=6.$

31. $3x^2+165=16xy,$
$7xy+3y^2=132.$

32. $3x^2+xy+y^2=15,$
$31xy-3x^2-5y^2=45.$

33. $x^2+y^2-3=3xy,$
$2x^2-6+y^2=0.$

34. $x^4+y^4=706,$
$x+y=8.$

35. $x^4+y^4=272,$
$x-y=2.$

36. $x^5-y^5=992,$
$x-y=2.$

37. $x^2y^4-6xy^2=-9,$
$xy-y=2.$

38. $2x^3+2y^3=9xy,$
$x+y=3.$

39. $x+y=1072,$
$x^{\frac{1}{3}}+y^{\frac{1}{3}}=16.$

40. $xy^{\frac{1}{2}}+yx^{\frac{1}{2}}=20,$
$x^{\frac{3}{2}}+y^{\frac{3}{2}}=65.$

41. $x^{\frac{1}{2}}+y^{\frac{1}{2}}=5,$
$6(x^{-\frac{1}{2}}+y^{-\frac{1}{2}})=5.$

42. $\sqrt{\frac{x}{y}}+\sqrt{\frac{y}{x}}=\frac{10}{3},$
$x+y=10.$

43. $\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}+\frac{\sqrt{x}+\sqrt{y}}{\sqrt{x}-\sqrt{y}}=\frac{17}{4},$
$x^2+y^2=706.$

44. $4x^2+5y=6+20xy-25y^2+2x,$
$7x-11y=17.$

45. $x^2y^2+400=41xy,$
$y^2+4x^2=5xy.$

46. $9x^2+33x-12=12xy-4y^2+22y,$
$x^2-xy=18.$

47. $xy+ab=2ax,$
$x^2y^2+a^2b^2=2b^2y^2.$

48. $(x^2-y^2)(x-y)=16xy,$
$(x^4-y^4)(x^2-y^2)=640x^2y^2.$

49. $2x^2-xy+y^2=2y,$
$2x^2+4xy=5y.$

50. $x^2+y^2-z^2=21,$
$3xz+3yz-2xy=18,$
$x+y-z=5.$

51. $x-y-z=2,$
$x^2+y^2-z^2=22,$
$xy=5.$

52. $x^2+xy+xz=18,$
$y^2+yz+yx=-12,$
$z^2+zx+zy=30.$

CHAPTER XXIX.

Problems leading to Quadratic Equations.

306. We shall now discuss some problems which give rise to quadratic equations.

Ex. 1. A train travels 300 miles at a uniform rate; if the rate had been 5 miles an hour more, the journey would have taken two hours less: find the rate of the train.

Suppose the train travels at the rate of x miles per hour, then the time occupied is $\frac{300}{x}$ hours.

On the other supposition the time is $\frac{300}{x+5}$ hours;

$$\therefore \frac{300}{x+5} = \frac{300}{x} - 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1);$$

whence $$x^2 + 5x - 750 = 0,$$

or $$(x+30)(x-25) = 0,$$

$$\therefore x = 25, \text{ or } -30.$$

Hence the train travels 25 miles per hour, the negative value being inadmissible.

It will frequently happen that the algebraic statement of the question leads to a result which does not apply to the actual problem we are discussing. But such results can sometimes be explained by a suitable modification of the conditions of the question. In the present case we may explain the negative solution as follows:

Since the values $x = 25$ and -30 satisfy the equation (1), if we write $-x$ for x, the resulting equation,

$$\frac{300}{-x+5} = \frac{300}{-x} - 2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

will be satisfied by the values $x = -25$ and 30. Now, by changing signs throughout, equation (2) becomes $\frac{300}{x-5} = \frac{300}{x} + 2$;

and this is the algebraic statement of the following question:

A train travels 300 miles at a uniform rate; if the rate had been 5 miles an hour *less*, the journey would have taken two hours *more*; find the rate of the train. The rate is 30 miles an hour.

Ex. 2. A person, selling a horse for \$72, finds that his loss per cent is one-eighth of the number of dollars that he paid for the horse: what was the cost price?

Suppose that the cost price of the horse is x dollars; then the loss on \$100 is $\$\frac{x}{8}$.

Hence the loss on $\$x$ is $x \times \frac{x}{800}$, or $\frac{x^2}{800}$ dollars;

$\therefore$ the selling price is $x - \frac{x^2}{800}$ dollars.

Hence
$$x - \frac{x^2}{800} = 72,$$
or
$$x^2 - 800x + 57600 = 0;$$
that is,
$$(x-80)(x-720) = 0;$$
$$\therefore x = 80, \text{ or } 720;$$

and each of these values will be found to satisfy the conditions of the problem. Thus the cost is either \$80, or \$720.

Ex. 3. A cistern can be filled by two pipes in $33\frac{1}{3}$ minutes; if the larger pipe takes 15 minutes less than the smaller to fill the cistern, find in what time it will be filled by each pipe singly.

Suppose that the two pipes running singly would fill the cistern in x and $x-15$ minutes. When running together they will fill $\left(\frac{1}{x} + \frac{1}{x-15}\right)$ of the cistern in one minute. But they fill $\frac{1}{33\frac{1}{3}}$, or $\frac{3}{100}$ of the cistern in one minute; hence

$$\frac{1}{x} + \frac{1}{x-15} = \frac{3}{100},$$
$$100(2x-15) = 3x(x-15),$$
$$3x^2 - 245x + 1500 = 0,$$
$$(x-75)(3x-20) = 0;$$
$$\therefore x = 75, \text{ or } 6\tfrac{2}{3}.$$

Thus the smaller pipe takes 75 minutes, the larger 60 minutes. The other solution, $6\frac{2}{3}$, is inadmissible.

Ex. 4. The small wheel of a bicycle makes 135 revolutions more than the large wheel in a distance of 260 yards; if the circumference of each were one foot more, the small wheel would make 27 revolutions more than the large wheel in a distance of 70 yards: find the circumference of each wheel.

Suppose the small wheel to be x feet, and the large wheel y feet in circumference.

In a distance of 260 yards, the two wheels make $\frac{780}{x}$ and $\frac{780}{y}$ revolutions respectively.

Hence
$$\frac{780}{x} - \frac{780}{y} = 135,$$

or
$$\frac{1}{x} - \frac{1}{y} = \frac{9}{52}. \quad \ldots \ldots \quad (1).$$

Similarly, from the second condition, we obtain
$$\frac{210}{x+1} - \frac{210}{y+1} = 27,$$

or
$$\frac{1}{x+1} - \frac{1}{y+1} = \frac{9}{70}. \quad \ldots \ldots \quad (2).$$

From (1),
$$x = \frac{52\,y}{52 + 9\,y};$$

whence
$$x + 1 = \frac{61\,y + 52}{9\,y + 52}.$$

Substituting in (2),
$$\frac{9\,y + 52}{61\,y + 52} - \frac{1}{y+1} = \frac{9}{70},$$

$$70 \times 9\,y^2 = 9(61\,y + 52)(y + 1),$$

$$9\,y^2 - 113\,y - 52 = 0,$$

$$(y - 13)(9\,y + 4) = 0;$$

$$\therefore\ y = 13, \text{ or } -\tfrac{4}{9}.$$

Putting $y = 13$, we find that $x = 4$. The other value of y is inadmissible; hence the small wheel is 4 feet, the large wheel 13 feet in circumference.

Ex. **5.** On a river, there are two towns 24 miles apart. By rowing one half of the distance, and walking the other half, a man performs the journey down stream in 5 hours, and up stream in 7 hours. Had there been no current, each journey would have taken $5\frac{2}{3}$ hours. Find the rate of his walking, and rowing, and the rate of the stream.

Suppose that the man walks x miles per hour, rows y miles per hour, and that the stream flows at the rate of z miles per hour.

With the current the man rows $y + z$ miles, and against the current $y - z$ miles per hour.

Hence we have the following equations:

$$\frac{12}{x}+\frac{12}{y+z}=5 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (1),$$

$$\frac{12}{x}+\frac{12}{y-z}=7 \quad . \; . \; . \; . \; . \; . \; . \; . \quad (2),$$

$$\frac{12}{x}+\frac{12}{y}=5\tfrac{2}{3} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (3).$$

From (1) and (3), by subtraction, $\frac{1}{y}-\frac{1}{y+z}=\frac{1}{18}$ . . . . . (4).

Similarly, from (2) and (3), $\frac{1}{y-z}-\frac{1}{y}=\frac{1}{9}$ . . . . . (5).

From (4) $18z=y(y+z)$ . . . . . . (6);

and from (5) $9z=y(y-z)$ . . . . . . (7).

From (6) and (7), by division, $2=\frac{y+z}{y-z}$;

whence $y=3z$;

$\therefore$ from (4) $z=1\frac{1}{2}$; and hence $y=4\frac{1}{2}$, $x=4$.

Thus the rates of walking and rowing are 4 miles and $4\frac{1}{2}$ miles per hour respectively; and the stream flows at the rate of $1\frac{1}{2}$ miles per hour.

EXAMPLES XXIX.

1. Find a number whose square diminished by 119 is equal to ten times the excess of the number over 8.

2. A man is five times as old as his son, and the sum of the squares of their ages is equal to 2106: find their ages.

3. The sum of the reciprocals of two consecutive numbers is $\frac{15}{56}$: find them.

4. Find a number which when increased by 17 is equal to 60 times the reciprocal of the number.

5. Find two numbers whose sum is 9 times their difference, and the difference of whose squares is 81.

6. The sum of a number and its square is nine times the next higher number: find it.

7. If a train travelled 5 miles an hour faster, it would take one hour less to travel 210 miles: what time does it take?

8. Find two numbers the sum of whose squares is 74, and whose sum is 12.

9. The perimeter of a rectangular field is 500 yards, and its area is 14400 square yards: find the length of the sides.

10. The perimeter of one square exceeds that of another by 100 feet; and the area of the larger square exceeds three times the area of the smaller by 325 square feet: find the length of their sides.

11. A cistern can be filled by two pipes running together in $22\frac{1}{2}$ minutes; the larger pipe would fill the cistern in 24 minutes less than the smaller one: find the time taken by each.

12. A man travels 108 miles, and finds that he could have made the journey in $4\frac{1}{2}$ hours less had he travelled 2 miles an hour faster: at what rate did he travel?

13. I buy a number of foot-balls for \$100; had they cost a dollar apiece less, I should have had five more for the money: find the cost of each.

14. A boy was sent for 40 cents' worth of eggs. He broke 4 on his way home, and the cost therefore was at the rate of 3 cents more than the market price for 6. How many did he buy?

15. What are the two parts of 20 whose product is equal to 24 times their difference?

16. A lawn 50 feet long and 34 feet broad has a path of uniform width round it; if the area of the path is 540 square feet, find its width.

17. A hall can be paved with 200 square tiles of a certain size; if each tile were one inch longer each way it would take 128 tiles: find the length of each tile.

18. In the centre of a square garden is a square lawn; outside this is a gravel walk 4 feet wide, and then a flower border 6 feet wide. If the flower border and lawn together contain 721 square feet, find the area of the lawn.

19. By lowering the price of apples and selling them one cent a dozen cheaper, an applewoman finds that she can sell 60 more than she used to do for 60 cents. At what price per dozen did she sell them at first?

20. Two rectangles contain the same area, 480 square yards. The difference of their lengths is 10 yards, and of their breadths 4 yards: find their sides.

21. There is a number between 10 and 100; when multiplied by the digit on the left the product is 280; if the sum of the digits be multiplied by the same digit, the product is 55: find it.

22. A farmer having sold, at \$75 each, horses which cost him x dollars apiece, finds that he has realized x per cent profit on his outlay: find x.

23. If a carriage wheel $14\frac{2}{3}$ ft. in circumference takes one second more to revolve, the rate of the carriage per hour will be $2\frac{2}{3}$ miles less: how fast is the carriage travelling?

24. A merchant bought a number of yards of cloth for \$100; he kept 5 yards and sold the rest at \$2 per yard more than he gave, and received \$20 more than he originally spent: how many yards did he buy?

25. A broker bought as many shares of stock as cost him \$1875; he reserved 15, and sold the remainder for \$1740, gaining \$4 a share on their cost price. How many shares did he buy?

26. A and B are two stations 300 miles apart. Two trains start simultaneously from A and B, each to the opposite station. The train from A reaches B nine hours, the train from B reaches A four hours after they meet: find the rate at which each train travels.

27. A train A starts to go from P to Q, two stations 240 miles apart, and travels uniformly. An hour later another train B starts from P, and after travelling for 2 hours, comes to a point that A had passed 45 minutes previously. The pace of B is now increased by 5 miles an hour, and it overtakes A just on entering Q. Find the rates at which they started.

28. A cask P is filled with 50 gallons of water, and a cask Q with 40 gallons of brandy; x gallons are drawn from each cask, mixed and replaced; and the same operation is repeated. Find x when there are $8\frac{7}{8}$ gallons of brandy in P after the second replacement.

29. Two farmers A and B have 30 cows between them; they sell at different prices, but each receives the same sum. If A had sold his at B's price, he would have received \$320; and if B had sold his at A's price, he would have received \$245. How many had each?

30. A man arrives at the railroad station nearest to his house $1\frac{1}{2}$ hours before the time at which he had ordered his carriage to meet him. He sets out at once to walk at the rate of 4 miles an hour, and, meeting his carriage when it had travelled 8 miles, reaches home exactly 1 hour earlier than he had originally expected. How far is his house from the station, and at what rate was his carriage driven?

CHAPTER XXX.

Theory of Quadratic Equations.

MISCELLANEOUS THEOREMS.

307. In Chapter XXVI. it was shown that after suitable reduction every quadratic equation may be written in the form

$$ax^2 + bx + c = 0 \quad . \quad . \quad . \quad . \quad . \quad (1),$$

and that the solution of the equation is

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

We shall now prove some important propositions connected with the roots and coefficients of all equations of which (1) is the type.

NUMBER OF THE ROOTS.

308. A quadratic equation cannot have more than two roots.

For, if possible, let the equation $ax^2 + bx + c = 0$ have three *different* roots r_1, r_2, r_3. Then since each of these values must satisfy the equation, we have

$$ar_1^2 + br_1 + c = 0 \quad . \quad . \quad . \quad . \quad . \quad (1),$$
$$ar_2^2 + br_2 + c = 0 \quad . \quad . \quad . \quad . \quad . \quad (2),$$
$$ar_3^2 + br_3 + c = 0 \quad . \quad . \quad . \quad . \quad . \quad (3).$$

From (1) and (2), by subtraction,

$$a(r_1^2 - r_2^2) + b(r_1 - r_2) = 0;$$

divide out by $r_1 - r_2$, which, by hypothesis, is not zero; then

$$a(r_1 + r_2) + b = 0.$$

Similarly from (2) and (3),

$$a(r_2+r_3)+b=0;$$

$\therefore$ by subtraction, $\qquad a(r_1-r_3)=0;$

which is impossible, since, by hypothesis, a is not zero, and r_1 is not equal to r_3. Hence there cannot be three different roots.

309. The terms 'unreal,' 'imaginary,' and 'impossible' are all used in the same sense; namely, to denote expressions which involve the square root of a negative quantity, such as

$$\sqrt{-1},\ \sqrt{-3},\ \sqrt{-a}.$$

It is important that the student should clearly distinguish between the terms **real** and **rational, imaginary** and **irrational.** Thus $\sqrt{25}$ or 5, $3\frac{1}{2}$, $-\frac{5}{6}$ are *rational* and *real;* $\sqrt{7}$ is *irrational* but *real;* while $\sqrt{-7}$ is *irrational* and also *imaginary.*

CHARACTER OF THE ROOTS.

310. In Art. 307 denote the two roots in (2) by r_1 and r_2,

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a},\quad r_2=\frac{-b-\sqrt{b^2-4ac}}{2a};$$

then we have the following results:

(1) If b^2-4ac, the quantity under the radical, is *positive,* the roots are **real and unequal.**

(2) If b^2-4ac is *zero,* the roots are **real and equal,** each reducing in this case to $-\dfrac{b}{2a}$.

(3) If b^2-4ac is *negative,* the roots are **imaginary and unequal.**

(4) If b^2-4ac is a *perfect square,* the roots are **rational and unequal.**

By applying these tests the nature of the roots of any quadratic may be determined without solving the equation.

Ex. 1. Show that the equation $2x^2-6x+7=0$ cannot be satisfied by any real values of x.

Here $a=2$, $b=-6$, $c=7$; so that

$$b^2-4ac=(-6)^2-4\cdot 2\cdot 7=-20.$$

Therefore the roots are imaginary.

Ex. 2. For what value of k will the equation $3x^2-6x+k=0$ have equal roots?

The condition for equal roots gives

$$(-6)^2-4\cdot 3\cdot k=0,$$

whence $$k=3.$$

Ex. 3. Show that the roots of the equation

$$x^2-2ax+a^2-b^2+2bc-c^2=0$$ are rational.

The roots will be rational provided $(-2a)^2-4(a^2-b^2+2bc-c^2)$ is a perfect square. But this expression reduces to $4(b^2-2bc+c^2)$ or $4(b-c)^2$. Hence the roots are rational.

RELATIONS OF ROOTS AND COEFFICIENTS.

311. Since $r_1=\dfrac{-b+\sqrt{b^2-4ac}}{2a}$, $r_2=\dfrac{-b-\sqrt{b^2-4ac}}{2a}$,

we have by addition

$$r_1+r_2=\frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a}=-\frac{b}{a}\quad . \quad (1),$$

and by multiplication we have

$$r_1r_2=\frac{(-b+\sqrt{b^2-4ac})(-b-\sqrt{b^2-4ac})}{4a^2}$$

$$=(-b)^2-(b^2-4ac)=\frac{4ac}{4a^2}=\frac{c}{a}\quad . \quad . \quad . \quad . \quad (2).$$

By writing the equation in the form $x^2+\dfrac{b}{a}x+\dfrac{c}{a}=0$, these results may also be expressed as follows:

In a quadratic equation *where the coefficient of the first term is unity*,

(i.) the **sum** of the roots is equal to the coefficient of x with its sign changed;

(ii.) the **product** of the roots is equal to the third term.

NOTE. In any equation the term which does not contain the unknown quantity is frequently called *the absolute term.*

FORMATION OF EQUATIONS WITH GIVEN ROOTS.

312. Since $-\frac{b}{a} = r_1 + r_2$, and $\frac{c}{a} = r_1 r_2$,

the equation $x^2 + \frac{b}{a}x + \frac{c}{a}$ may be written

$$x^2 - (r_1 + r_2)x + r_1 r_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Hence any quadratic may also be expressed in the form

$$x^2 - (\text{sum of roots})\,x + \text{product of roots} = 0 \quad . \quad (2).$$

Again, from (1) we have

$$(x - r_1)(x - r_2) = 0 \quad . \quad . \quad . \quad . \quad . \quad (3).$$

We may now form an equation with given roots.

Ex. **1.** Form the equation whose roots are 3 and -2.

The equation is $(x-3)(x+2) = 0$,

or $x^2 - x - 6 = 0$.

Ex. **2.** Form the equation whose roots are $\frac{3}{7}$ and $-\frac{4}{5}$.

The equation is $(x - \frac{3}{7})(x + \frac{4}{5}) = 0$;

that is, $(7x-3)(5x+4) = 0$,

or $35x^2 + 13x - 12 = 0$.

When the roots are irrational it is easier to use the following method.

Ex. **3.** Form the equation whose roots are $2+\sqrt{3}$ and $2-\sqrt{3}$.

We have sum of roots $= 4$,

product of roots $= 1$;

$\therefore$ the equation is $x^2 - 4x + 1 = 0$,

by using formula (2) of the present article.

313. The results of Art. 311 are most important, and they are generally sufficient to solve problems connected with the roots of quadratics. In such questions *the roots should never be considered singly,* but use should be made of the relations obtained by writing down the sum of the roots, and their product, in terms of the coefficients of the equation.

Ex. 1. If a and b are the roots of $x^2 - px + q = 0$, find the value of (1) $a^2 + b^2$, (2) $a^3 + b^3$.

We have
$$a + b = p,$$
$$ab = q.$$
$$\therefore\ a^2 + b^2 = (a + b)^2 - 2ab = p^2 - 2q.$$

Again,
$$a^3 + b^3 = (a + b)(a^2 + b^2 - ab)$$
$$= p\{(a + b)^2 - 3ab\} = p(p^2 - 3q).$$

Ex. 2. If a, b are the roots of the equation $lx^2 + mx + n = 0$, find the equation whose roots are $\frac{a}{b}$, $\frac{b}{a}$.

We have
$$\text{sum of roots} = \frac{a}{b} + \frac{b}{a} = \frac{a^2 + b^2}{ab},$$
$$\text{product of roots} = \frac{a}{b} \times \frac{b}{a} = 1;$$

$\therefore$ by Art. 312 the required equation is
$$x^2 - \left(\frac{a^2 + b^2}{ab}\right)x + 1 = 0,$$
or
$$abx^2 - (a^2 + b^2)x + ab = 0.$$

As in the last example $a^2 + b^2 = \frac{m^2 - 2nl}{l^2}$, and $ab = \frac{n}{l}$,

$\therefore$ the equation is $\quad \frac{n}{l}x^2 - \frac{m^2 - 2nl}{l^2}x + \frac{n}{l} = 0,$

or
$$nlx^2 - (m^2 - 2nl)x + nl = 0.$$

Ex. 3. Find the condition that the roots of the equation
$$ax^2 + bx + c = 0$$
should be (1) equal in magnitude and opposite in sign, (2) reciprocals.

The roots will be equal in magnitude and opposite in sign if their sum is zero; therefore $-\frac{b}{a} = 0$, or $b = 0$.

Again, the roots will be reciprocals when their product is unity; therefore $\frac{c}{a} = 1$, or $c = a$.

Ex. 4. Find the relation which must subsist between the coefficients of the equation $px^2 + qx + r = 0$ when one root is three times the other.

We have
$$a + b = -\frac{q}{p},\ ab = \frac{r}{p};$$
but since $a = 3b$, we obtain by substitution
$$4b = -\frac{q}{p},\ 3b^2 = \frac{r}{p}.$$

From the first of these equations $b^2 = \frac{q^2}{16p^2}$, and from the second $b^2 = \frac{r}{3p}$.

$$\therefore \frac{q^2}{16p^2} = \frac{r}{3p},$$

or

$$3q^2 = 16pr,$$

which is the required condition.

314. The following example illustrates a useful application of the results proved in Art. 310.

Ex. If x is a real quantity, prove that the expression $\frac{x^2+2x-11}{2(x-3)}$ can have all numerical values except such as lie between 2 and 6.

Let the given expression be represented by y, so that

$$\frac{x^2+2x-11}{2(x-3)} = y;$$

then multiplying across and transposing, we have

$$x^2 + 2x(1-y) + 6y - 11 = 0.$$

This is a quadratic equation, and if x is to have real values, $4(1-y)^2 - 4(6y-11)$ must be positive; or simplifying and dividing by 4, $y^2 - 8y + 12$ must be positive; that is, $(y-6)(y-2)$ must be positive. Hence the factors of this product must be both positive or both negative. In the former case y is greater than 6; in the latter y is less than 2. Therefore y cannot lie between 2 and 6, but may have any other value.

In this example it will be noticed that the *expression* $y^2 - 8y + 12$ is positive so long as y does not lie between the roots of the corresponding quadratic equation

$$y^2 - 8y + 12 = 0.$$

This is a particular case of the general proposition investigated in the next article.

315. For all real values of x the expression $ax^2 + bx + c$ has the same sign as a, except when the roots of the equation $ax^2 + bx + c = 0$ are real and unequal, and x lies between them.

Case I. Suppose that the roots of the equation

$$ax^2 + bx + c = 0$$

are real; denote them by r_1 and r_2, and let r_1 be the greater.

Then $$ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$
$$= a\{x^2 - (r_1 + r_2)x + r_1r_2\} \quad \text{[Art. 311.]}$$
$$= a(x - r_1)(x - r_2).$$

Now if x is greater than r_1 or less than r_2, the factors $x - r_1$, $x - r_2$ are either both positive or both negative; therefore the expression $(x - r_1)(x - r_2)$ is positive, and ax^2+bx+c has the same sign as r_1. But if x lies between r_1 and r_2, the expression $(x - r_1)(x - r_2)$ is negative, and the sign of $ax^2 + bx + c$ is opposite to that of a.

Case II. If r_1 and r_2 are equal, then
$$ax^2 + bx + c = a(x - r_1)^2,$$
and $(x - r_1)^2$ is positive for all real values of x; hence $ax^2 + bx + c$ has the same sign as a.

Case III. Suppose that the equation $ax^2 + bx + c = 0$ has imaginary roots; then
$$ax^2 + bx + c = a\left\{x^2 + \frac{b}{a}x + \frac{c}{a}\right\}$$
$$= a\left\{\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}\right\};$$
but since $b^2 - 4ac$ is negative [Art. 310], the expression
$$\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}$$
is positive for all real values of x; therefore $ax^2 + bx + c$ has the same sign as a.

EXAMPLES XXX. a.

Find (without actual solution) the nature of the roots of the following equations:

1. $x^2 + x - 870 = 0$.
2. $8 + 6x = 5x^2$.
3. $\frac{1}{2}x^2 = 14 - 3x^2$.
4. $x^2 + 7 = 4x$.
5. $2x = x^2 + 5$.
6. $(x+2)^2 = 4x + 15$.

Form the equations whose roots are

7. $5, \; -3$.
8. $-9, \; -11$.
9. $a + b, \; a - b$.
10. $\frac{3}{2}, \; \frac{5}{6}$.
11. $\frac{2}{3}a, \; -\frac{4}{5}a$.
12. $0, \; \frac{7}{8}$.

13. If the equation $x^2+2(1+k)x+k^2=0$ has equal roots, what is the value of k?

14. Prove that the equation $3mx^2-(2m+3n)x+2n=0$ has rational roots.

15. Without solving the equation $3x^2-4x-1=0$, find the sum, the difference, and the sum of the squares of the roots.

16. Show that the roots of $a(x^2-1)=(b-c)x$ are always real.

Form the equations whose roots are

17. $3+\sqrt{5},\ 3-\sqrt{5}$. **18.** $-2+\sqrt{3},\ -2-\sqrt{3}$. **19.** $-\frac{a}{5},\ \frac{b}{6}$.

20. $\frac{1}{2}(4\pm\sqrt{7})$. **21.** $\frac{a+b}{a-b},\ \frac{a-b}{a+b}$. **22.** $\frac{a}{2b},\ \frac{b}{2a}$.

If a, b are the roots of the equation $px^2+qx+r=0$, find the values of

23. a^2+b^2. **25.** a^2b+ab^2. **27.** $a^5b^2+a^2b^5$.

24. $(a-b)^2$. **26.** a^4+b^4. **28.** $\frac{a^2}{b}+\frac{b^2}{a}$.

29. If a, b are the roots of $x^2-px+q=0$, and a^3, b^3 the roots of $x^2-Px+Q=0$, find P and Q in terms of p and q.

30. If a, b are the roots of $x^2-ax+b=0$, find the equation whose roots are $\frac{a}{b^2},\ \frac{b}{a^2}$.

31. Find the condition that one root of the equation $ax^2+bx+c=0$ may be double the other.

32. Form an equation whose roots shall be the cubes of the roots of the equation $2x(x-a)=a^2$.

33. Prove that the roots of the equation

$$(a+b)x^2-(a+b+c)x+\frac{c}{2}=0$$

are always real.

34. Show that $(a+b+c)x^2-2(a+b)x+(a+b-c)=0$ has rational roots.

35. Show that if x is real the expression $\frac{x^2-15}{2x-8}$ cannot lie between 3 and 5.

36. If x is real, prove that $\frac{3x^2+2}{x^2-2x-1}$ can have all values except such as lie between 2 and $-\frac{3}{2}$.

MISCELLANEOUS THEOREMS.

316. The Remainder Theorem. *If any algebraical expression* $x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \cdots + p_{n-1}x + p_n$ *be divided by* $x - a$, *the remainder will be*

$$a^n + p_1a^{n-1} + p_2a^{n-2} + p_3a^{n-3} + \cdots + p_{n-1}a + p_n.$$

Divide the given expression by $x - a$ till a remainder is obtained which does not involve x. Let Q be the quotient, and R the remainder; then

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = Q(x - a) + R.$$

Since R does not contain x, it will remain unaltered whatever value we give to x.

Put $x = a$, then

$$a^n + p_1a^{n-1} + p_2a^{n-2} + \cdots + p_{n-1}a + p_n = Q \times 0 + R,$$

$$\therefore\ R = a^n + p_1a^{n-1} + p_2a^{n-2} + \cdots + p_{n-1}a + p_n;$$

which proves the proposition.

From this it appears that when an algebraic expression with integral exponents is divided by $x - a$, the remainder can be obtained at once by writing a in the place of x in the given expression.

Ex. The remainder when $x^4 - 2x^3 + x - 7$ is divided by $x + 2$ is

$$(-2)^4 - 2(-2)^3 + (-2) - 7;$$

that is, $16 + 16 - 2 - 7$, or 23.

317. In the preceding article the remainder is zero when the given expression is exactly divisible by $x - a$; hence we deduce:

The Factor Theorem. *If any rational and integral expression containing* x *becomes equal to* 0 *when* a *is written for* x, *it is exactly divisible by* $x - a$. (See Art. 105.)

318. *To find the condition that* $x^2 + px + q$ *may be a perfect square.*

It must be evident that any such general expression cannot be a perfect square unless some particular relation subsists between the coefficients p and q. To find the necessary connection between p and q is the object of the present question.

Using the ordinary rule for square root, we have

$$\begin{array}{r|l}
 & x^2 + px + q\,\big(\,x + \dfrac{p}{2} \\
 & x^2 \\ \hline
2x + \dfrac{p}{2} & px + q \\
 & px + \dfrac{p^2}{4} \\ \hline
 & q - \dfrac{p^2}{4}
\end{array}$$

If therefore x^2+px+q be a perfect square, the remainder, $q-\dfrac{p^2}{4}$, must be zero. Hence the condition is determined by placing this remainder equal to zero and solving the resulting equation.

319. Symmetry. An expression is said to be *symmetrical* with respect to the letters it contains when its value is unaltered by the interchange of any pair of them; thus $x+y+z$, $bc+ca+ab$, $x^3+y^3+z^3-xyz$ are symmetrical functions of the first, second, and third degrees respectively.

It is worthy of notice that the only symmetrical expression of the first degree in x, y, z is of the form $M(x+y+z)$, where M is independent of x, y, z.

320. It easily follows from the definition that the sum, difference, product, and quotient of any two symmetrical expressions must also be symmetrical expressions. The recognition of this principle is of great use in checking the accuracy of algebraic work, and in some cases enables us to dispense with much of the labor of calculation. In the following examples we shall assume as true a principle which will be demonstrated in Chap. XLII.

Ex. 1. Find the expansion of $(x+y+z)^3$. We know that the expansion must be a homogeneous expression of three dimensions, and therefore of the form $x^3 + y^3 + z^3 + A(x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2) + Bxyz$, where A and B are quantities independent of x, y, z.

Put $z = 0$, then A, the coefficient of x^2y, is equal to 3, the coefficient of x^2y in the expansion of $(x + y)^3$.

Put $x = y = z = 1$, and we get $27 = 3 + (3 \times 6) + B$; whence $B = 6$.

Thus $(x + y + z)^3$

$$= x^3 + y^3 + z^3 + 3x^2y + 3xy^2 + 3y^2z + 3yz^2 + 3z^2x + 3zx^2 + 6xyz.$$

Ex. 2. Find the factors of

$$(b^3 + c^3)(b - c) + (c^3 + a^3)(c - a) + (a^3 + b^3)(a - b).$$

Denote the expression by E; then E is an expression involving a, which vanishes when $a = b$, and therefore contains $a - b$ as a factor [Art. 317]. Similarly it contains the factors $b - c$ and $c - a$; thus E contains $(b - c)(c - a)(a - b)$ as a factor.

Also since E is of the fourth degree, the remaining factor must be of the first degree; and since it is a symmetrical expression involving a, b, c, it must be of the form $m(a + b + c)$. [Art. 319.]

$$\therefore\ E = m(b - c)(c - a)(a - b)(a + b + c).$$

To obtain m we may give to a, b, c any values that we find most convenient; thus by putting $a = 0$, $b = 1$, $c = 2$, we find $m = 1$, and we have the required result.

Note. For further information on the subject of Symmetry, the reader may consult Hall and Knight's Higher Algebra, Chap. xxxiv.

EXAMPLES XXX. b.

Without actual division find the remainder when

1. $x^5 - 5x^2 + 5$ is divided by $x - 5$.

2. $3x^5 + 11x^4 + 90x^2 - 19x + 53$ is divided by $x + 5$.

3. $x^3 - 7x^2a + 8xa^2 + 15a^3$ is divided by $x + 2a$.

Without actual division show that

4. $32x^{10} - 33x^5 + 1$ is divisible by $x - 1$.

5. $3x^4 + 5x^3 - 13x^2 - 20x + 4$ is divisible by $x^2 - 4$.

6. $x^4 + 4x^3 - 5x^2 - 36x - 36$ is divisible by $x^2 - x - 6$.

Resolve into factors:

7. $x^3 - 6x^2 + 11x - 6$.

8. $x^3 - 5x^2 - 2x + 24$.

9. $x^3 + 9x^2 + 26x + 24$.

10. $x^3 - x^2 - 41x + 105$.

11. $x^3 - 39x + 70$.

12. $x^3 - 8x^2 - 31x - 22$.

13. $6x^3 + 7x^2 - x - 2$.

14. $6x^3 + x^2 - 19x + 6$.

Find the values of x which will make each of the following expressions a perfect square:

15. $x^4 + 6x^3 + 13x^2 + 13x - 1.$ **16.** $x^4 + 6x^3 + 11x^2 + 3x + 31.$

17. $x^4 - 2ax^3 + (a^2 + 2b)x^2 - 3abx + 2b^2.$

18. $4p^2x^4 - 4pqx^3 + (q^2 + 2p^2)x^2 - 5pqx + \frac{p^2}{2}.$

19. $\frac{a^2x^6}{9} - \frac{abx^4}{2} + \frac{2acx^3}{3} + \frac{9b^2x^2}{16} - \frac{5bcx}{2} + 6c^2.$

20. $x^4 + 2ax^3 + 3a^2x^2 + cx + d.$

Find the values of x which will make each of the following expressions a perfect cube:

21. $8x^3 - 36x^2 + 56x - 39.$ **22.** $\frac{x^6}{27} - \frac{a^2x^4}{3} + 4a^4x^2 - 28a^6.$

23. $m^3x^6 - 9m^2nx^4 + 39mn^2x^2 - 51n^3.$

24. If n be any positive integer, prove that $5^{2n} - 1$ is always divisible by 24.

Find the factors of

25. $a(b-c)^3 + b(c-a)^3 + c(a-b)^3.$

26. $a(b-c)^2 + b(c-a)^2 + c(a-b)^2 + 8abc.$

CHAPTER XXXI.

Indeterminate Equations of the First Degree.

321. In Art. 167 we saw that if the number of unknown quantities is greater than the number of independent equations, there will be an unlimited number of solutions, and the equations will be *indeterminate*. By introducing conditions, however, we can limit the number of solutions. When *positive integral values* of the unknown quantities are required, the equations are called **simple indeterminate equations.**

The introduction of this restriction enables us to express the solutions in a very simple form.

Ex. 1. Solve $7x + 12y = 220$ in positive integers.

Transpose and divide by the smaller coefficient; thus,

$$x = 31 - y + \frac{3 - 5y}{7};$$

$$\therefore\ x + y - 31 = \frac{3 - 5y}{7}.$$

Since x and y are to be integers, we must have

$$\frac{3 - 5y}{7} = \text{integer}.$$

Now multiplying the numerator *by such a number that the division of the coefficient of* y *may give a remainder of unity*, in this case 3, we have

$$\frac{9 - 15y}{7} = \text{integer};$$

that is,

$$1 - 2y + \frac{2 - y}{7} = \text{integer};$$

and therefore

$$\frac{2 - y}{7} = \text{integer}.$$

Let $$\frac{2-y}{7} = m, \text{ an integer};$$

then $$y = 2 - 7m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Substituting this value in the original equation, we obtain

$$7x + 24 - 84m = 220;$$

$$\therefore\ x = 28 + 12m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Equation (1) shows that m may be 0 or have *any negative integral value, but cannot have a positive integral value.*

Equation (2) shows in addition that m may be 0, but cannot have a *negative* integral value greater than 2. Thus the only *positive integral* values of x and y are obtained by placing $m = 0,\ -1,\ -2.$

The complete solution may be exhibited as follows:

$$\begin{aligned} m &= \ \ 0, \quad -1, \quad -2, \\ x &= 28, \quad 16, \quad 4, \\ y &= \ \ 2, \quad 9, \quad 16. \end{aligned}$$

Ex. **2**. Solve $5x - 14y = 11$ in positive integers . . . . (1).

Proceeding as in Example 1, we obtain

$$x = 2 + 2y + \frac{4y+1}{5};$$

$$\therefore\ x - 2y - 2 = \frac{4y+1}{5} = \text{integer}.$$

Now multiplying the numerator by 4, we obtain

$$\frac{16y+4}{5} = \text{integer};$$

that is, $$3y + \frac{y+4}{5} = \text{integer}.$$

Let $$\frac{y+4}{5} = m, \text{ an integer};$$

and from (1), $$\therefore\ \left.\begin{aligned} y &= \ \ 5m - 4, \\ x &= 14m - 9. \end{aligned}\right\}$$

This is called the *general solution* of the equation, and by giving to m any positive integral value, we obtain an unlimited number of values for x and y: thus we have

$$\begin{aligned} m &= 1, \quad 2, \quad 3, \quad 4 \cdots \\ y &= 1, \quad 6, \quad 11, \quad 16 \cdots \\ x &= 5, \quad 19, \quad 33, \quad 47 \cdots \end{aligned}$$

From Examples 1 and 2 the student will see that there is a further limitation to the number of solutions according as the terms of the original equations are connected by + or —. If we have two equations involving three unknown quantities, we can easily combine them so as to eliminate one of the unknown quantities, and can then proceed as above.

Ex. 3. In how many ways can \$5 be paid in quarters and dimes?

Let $x =$ the number of quarters, y the number of dimes; then

$$\frac{x}{4} + \frac{y}{10} = 5,$$

or

$$5x + 2y = 100;$$

$$\therefore\ 2x + \frac{x}{2} + y = 50;$$

$$\therefore\ \frac{x}{2} = p, \text{ an integer};$$

$$\therefore\ x = 2p,$$

and

$$y = 50 - 5p.$$

Solutions are obtained by giving to p the values 1, 2, 3, ⋯, 9; and therefore the number of ways is 9. If, however, the sum be paid *either* in quarters or dimes, p may also have the values 0 and 10. If $p = 0$, then $x = 0$, and the sum is paid entirely in dimes; if $p = 10$, then $y = 0$, and the sum is paid entirely in quarters. Thus if zero values of x and y are admissible, the number of ways is 11.

EXAMPLES XXXI.

Solve in positive integers:

1. $3x + 8y = 103.$ **3.** $7x + 12y = 152.$ **5.** $23x + 25y = 915.$

2. $5x + 2y = 53.$ **4.** $13x + 11y = 414.$ **6.** $41x + 47y = 2191.$

Find the general solution in positive integers, and the least values of x and y which satisfy the equations:

7. $5x - 7y = 3.$ **9.** $8x - 21y = 33.$ **11.** $19y - 23x = 7.$

8. $6x - 13y = 1.$ **10.** $17y - 13x = 0.$ **12.** $77y - 30x = 295.$

13. A farmer spends \$752 in buying horses and cows; if each horse costs \$37, and each cow \$23, how many of each does he buy?

14. In how many ways can \$100 be paid in dollars and half-dollars, including zero solutions?

15. Find a number which, being divided by 39, gives a remainder 16, and, by 56, a remainder 27. How many such numbers are there?

CHAPTER XXXII.

INEQUALITIES.

322. Any quantity a is said to be greater than another quantity b when $a - b$ is positive; thus 2 is greater than -3, because $2 - (-3)$, or 5, is positive. Also b is said to be less than a when $b - a$ is negative; thus -5 is less than -2, because $-5 - (-2)$, or -3, is negative.

In accordance with this definition, *zero must be regarded as greater than any negative quantity.*

323. The statement in algebraic language that one expression is greater or less than another is called an **inequality.**

324. The **sign of inequality** is $>$, the opening being placed towards the greater quantity. Thus, $a > b$ is read "a is greater than b."

325. The **first and second members** are the expressions on the left and right, respectively, of the sign of inequality.

326. Inequalities **subsist in the same sense** when corresponding members in each are the greater or the less. Thus, the inequalities $a > b$ and $7 > 5$ are said to subsist in the same sense.

In the present chapter, we shall suppose (unless the contrary is directly stated) that the letters always denote real and positive quantities.

327. Inequality Unchanged. *An inequality will still hold after each side has been increased, diminished, multiplied, or divided by the same positive quantity.*

For, if $a > b$, then it is evident that

$$a + c > b + c;$$
$$a - c > b - c;$$
$$ac > bc;$$
$$\frac{a}{c} > \frac{b}{c}.$$

328. Term Transposed. *In an inequality any term may be transposed from one side to the other if its sign be changed.*

If $$a - c > b,$$
by adding c to each side,

$$a > b + c.$$

329. Members Transposed. *If the sides of an inequality be transposed, the sign of inequality must be reversed.*

For if $a > b$, then evidently $b < a$.

330. Signs Changed. *If the signs of all the terms of an inequality be changed, the sign of inequality must be reversed.*

When $a > b$, then $a - b$ is positive, and $b - a$ is negative; that is, $-a - (-b)$ is negative, and therefore

$$-a < -b.$$

331. Negative Multiplier. *If the sides of an inequality be multiplied by the same negative quantity, the sign of inequality must be reversed.*

For, if $a > b$, then $-a < -b$, and therefore

$$-ac < -bc.$$

332. Inequalities Combined. *If inequalities, subsisting in the same sense, be either added, or multiplied together, the results will be unequal in the same sense.*

For if $a_1 > b_1$, $a_2 > b_2$, $a_3 > b_3 \cdots a_m > b_m$, it is clear that

$$a_1 + a_2 + a_3 + \cdots + a_m > b_1 + b_2 + b_3 + \cdots + b_m;$$

and $$a_1 a_2 a_3 \cdots a_m > b_1 b_2 b_3 \cdots b_m.$$

333. It follows from the preceding article that if $a > b$,

then $$a^n > b^n,$$

and $$a^{-n} < b^{-n},$$

where n is any positive quantity.

334. The subtraction of two inequalities subsisting in the same sense does not *necessarily* give an inequality subsisting in the same sense.

335. The division of an inequality by another subsisting in the same sense does not *necessarily* give an inequality subsisting in the same sense.

The truth of these last statements is readily seen by considering the inequalities

$$5 > 4,$$

$$3 > 2.$$

Subtracting member for member would give $2 > 2$.

Dividing member by member would give $\frac{5}{3} > 2$.

Ex. 1. Find limit of x in the inequality

$$x - \frac{5}{3} < \frac{x}{5} + \frac{11}{15}.$$

Clearing of fractions, we have

$$15x - 25 > 3x + 11.$$

Transposing and combining,

$$12x > 36;$$

$$\therefore\ x > 3.$$

NOTE. The word "limit" is here used as meaning the range of values that x can have under the given conditions.

Ex. 2. If a, b, c denote positive quantities, prove that

$$a^2 + b^2 + c^2 > bc + ca + ab.$$

For $$b^2 + c^2 > 2bc,$$

$$c^2 + a^2 > 2ca$$

$$a^2 + b^2 > 2ab.$$

Whence by addition $a^2 + b^2 + c^2 > bc + ca + ab.$

Ex. 3. If x may have any real value, find which is the greater, x^3+1 or x^2+x.

$$x^3+1-(x^2+x)=x^3-x^2-(x-1)=(x^2-1)(x-1)$$
$$=(x-1)^2(x+1).$$

Now $(x-1)^2$ is positive, hence

$$x^3+1 > \text{ or } < x^2+x$$

according as $x+1$ is positive or negative; that is, according as $x >$ or < -1.

If $x=-1$, the inequality becomes an equality.

EXAMPLES XXXII.

Find limit of x in the following three inequalities:

1. $11x-\frac{46}{3}<\frac{5x}{3}+3\frac{1}{3}$.

2. $(x+2)(x+3)>(x-4)(x-5)$.

3. $bx+5ax-5ab>b^2$ when $a>b$.

4. Prove that $(ab+xy)(ax+by)>4abxy$.

5. Prove that $(b+c)(c+a)(a+b)>8abc$.

6. Show that the sum of any real positive quantity and its reciprocal is never less than 2.

7. If $a^2+b^2=1$, and $x^2+y^2=1$, show that $ax+by<1$.

8. If $a^2+b^2+c^2=1$, and $x^2+y^2+z^2=1$, show that
$$ax+by+cz<1.$$

9. Which is the greater $\frac{a+b}{2}$ or $\frac{2ab}{a+b}$?

10. Show that $(x^2y+y^2z+z^2x)(xy^2+yz^2+zx^2)>9x^2y^2z^2$.

11. Find which is the greater, $3ab^2$ or a^3+2b^3.

12. Prove that $a^3b+ab^3<a^4+b^4$.

13. Prove that $6abc<bc(b+c)+ca(c+a)+ab(a+b)$.

14. Show that $b^2c^2+c^2a^2+a^2b^2>abc(a+b+c)$.

15. Show that $2(a^3+b^3+c^3)>bc(b+c)+ca(c+a)+ab(a+b)$.

Note. For further information on the subject of Inequalities the reader may consult Hall and Knight's Higher Algebra, Chapter XIX.

MISCELLANEOUS EXAMPLES V.

1. If $a=-1$, $b=2$, $c=0$, $d=1$, $e=-3$, find the value of
$$\frac{a^3(d-c)-\sqrt{3ae}+ab}{d(c-a)-2ad^2+\sqrt[3]{4ab}}.$$

2. Simplify $[3(a-b+c)-(a-b)(b-c)+\{(a+b-c)(3-b)\}]$.

3. Solve $3x-4-\dfrac{4(7x-9)}{15}=\dfrac{4}{5}\left(6+\dfrac{x-1}{3}\right)$.

4. A man's age is four times the combined ages of his two sons, one of whom is three times as old as the other; in 24 years their combined ages will be 12 years less than their father's age: find their respective ages.

5. Simplify $\left(\dfrac{a^3b}{a^4b^{-3}}\right)^3 \div \left(\dfrac{\sqrt{a^{-\frac{1}{2}}} \times \sqrt[6]{b^{-4}}}{a^3b^2}\right)^4$.

6. Solve (i.) $3\sqrt{x}-1=\dfrac{5}{3\sqrt{x}+7}+6$.

(ii.) $\sqrt{8x+17}-\sqrt{2x}=\sqrt{2x+9}$.

7. Expand $(2a-3b^2)^5$.

8. Simplify $\dfrac{1}{2x+1}-\dfrac{1}{3(x+\frac{1}{2})}-\dfrac{1}{6x+3}$.

9. If 3 is added to the numerator a certain fraction is increased by $\frac{1}{6}$; if 3 is taken from the denominator the fraction reduces to $\frac{1}{3}$: find the fraction.

10. Find the value of (i.) $3\sqrt{243}+2\sqrt{\frac{27}{2}}+4\sqrt{75}-\sqrt{\frac{8}{3}}$.

(ii.) $\dfrac{\sqrt{3}+3\sqrt{2}}{2+\sqrt{6}}$.

11. Solve (i.) $\dfrac{3}{1-2x}+\dfrac{3}{1+2x}=16$.

(ii.) $\frac{4}{3}(x-2x^2)+\frac{3}{2}(1-2x)=5(\frac{1}{2}-x)$.

12. Find the limit of x in the inequality
$$(x-4)(x-5)>(x-2)(x-1).$$

13. The breadth of a rectangular space is 4 yards less than its length; the area of the space is 252 square yards: find the length of each side.

14. Solve (i.) $x^2+y^2=\frac{10}{9}$, $x+y=\frac{4}{3}$.

(ii.) $2x^2+xy=4$, $3xy+4y^2=22$.

15. Find the factors of (i.) $x^3+12x^2y-45xy^2$.

(ii.) $3x^2-31xy+56y^2$.

16. Simplify $\dfrac{x+1}{2x^3-4x^2}+\dfrac{x-1}{2x^3+4x^2}-\dfrac{1}{x^2-4}$.

17. Solve $\dfrac{x+4}{3}+\dfrac{x+y}{5}=6,\ \dfrac{2y+6}{y}-\dfrac{2x+4}{x}=\dfrac{5}{y}$.

18. Find two numbers whose sum is 22, and the sum of their squares is 250.

19. Simplify $\left[\sqrt[5]{\dfrac{b^2c^{-\frac{1}{2}}}{b^{-\frac{1}{2}}c^2}}\times\sqrt[3]{\dfrac{b^{-\frac{1}{2}}c}{c^{-\frac{1}{2}}b^{+1}}}\right]^{-4}$.

20. Solve (i.) $\dfrac{x-1}{\sqrt{x}-1}=3+\dfrac{\sqrt{x}+1}{2}$.

(ii.) $\sqrt{x^2+4x-4}+\sqrt{x^2+4x-10}=6$.

21. For what value of k will the equation $x^2+2(k+2)x+9k=0$ have equal roots?

22. Simplify the fractions:

(i.) $\dfrac{a-x}{a^2-ax-\dfrac{(a-x)^2}{1-\dfrac{a}{x}}}$. (ii.) $a-\dfrac{1}{b+\dfrac{1}{a+\dfrac{ab}{a-b}}}$.

23. B pays \$28 more rent for a field than A; he has three-fourths of an acre more and pays \$1.75 per acre more. C pays \$72.50 more than A; he has six and one-fourth acres more, but pays 25 cents per acre less: find the size of the fields.

24. Solve (i.) $\dfrac{x+10}{x-5}-\dfrac{10}{x}=\dfrac{11}{6}$.

(ii.) $\dfrac{2x}{x-4}+\dfrac{2x-5}{x-3}=\dfrac{25}{3}$.

25. Find the value of $\left(\dfrac{-1+\sqrt{-3}}{2}\right)^n+\left(\dfrac{-1-\sqrt{-3}}{2}\right)^n$ when $n=3$.

26. Rationalize the denominator of $\dfrac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$ and find a factor which will rationalize $\sqrt[3]{3}-\sqrt{2}$.

27. Find the square root of (i.) $11+4\sqrt{6}$.

(ii.) $-5+12\sqrt{-1}$.

28. Find the factors of (i.) $a^2-16-6ax+9x^2$.

(ii.) $343x^6-27y^3$.

29. Solve (i.) $x^3 + y^3 = 18\sqrt{2},\ x + y = 3\sqrt{2}.$

(ii.) $\frac{1}{x^3} + \frac{1}{y^3} = \frac{9}{8},\ \frac{1}{x} + \frac{1}{y} = \frac{3}{2}.$

30. A rectangular field is 100 yards wide. If it were reduced to a square field by cutting an oblong piece off one end, the ratio of the piece cut off to the remainder would be less by $\frac{1}{40}$ than the ratio of the remainder to the original field. Find the length of the field.

31. Extract the square root of

(i.) $4x^4 + 12x^3y + 13x^2y^2 + 6xy^3 + y^4.$

(ii.) $\frac{x^2}{a^2} - \frac{2x}{a} + 3 - \frac{2a}{x} + \frac{a^2}{x^2}.$

32. Solve (i.)
$$\begin{aligned} 2x + y + 3z &= 13, \\ x + 2y + 4z &= 17, \\ 4x + 3y + 2z &= 16. \end{aligned}$$

(ii.) $\frac{1}{1-x} + \frac{1}{\sqrt{x}+1} + \frac{1}{\sqrt{x}-1} = 0.$

33. Simplify the fractions

(i.) $\dfrac{1}{1-\dfrac{1}{2x}} \times \dfrac{1}{1-\dfrac{1}{1-x}} \div \dfrac{x}{2+\dfrac{1}{x-1}}.$

(ii.) $\dfrac{2x^3 - 7x^2y + 5xy^2 - y^3}{2x^3 + 5x^2y - 5xy^2 + y^3}.$

34. Two places, A and B, are 168 miles apart, and trains leave A for B and B for A simultaneously; they pass each other at the end of one hour and fifty-two minutes, and the first reaches B half an hour before the second reaches A. Find the speed of each train.

35. Solve (i.) $3x^2 + 4x + 2\sqrt{x^2 + x + 3} = 30 - x^2.$

(ii.) $x^4 + y^4 = 706,\ x + y = 8.$

36. Form the equation whose roots are the squares of the sum and of the difference of the roots of

$$2x^2 + 2(m+n)x + m^2 + n^2 = 0.$$

37. Employ the method of Arts. 319, 320 in showing that

$$(a+b)^5 - a^5 - b^5 = 5ab(a+b)(a^2 + ab + b^2).$$

38. Solve $xy(3x + y) = 10,\ 27x^3 + y^3 = 35.$

CHAPTER XXXIII.

Ratio, Proportion, and Variation.

336. Definition. **Ratio** is the relation which one quantity bears to another of the *same* kind, the comparison being made by considering what multiple, part, or parts, one quantity is of the other.

The ratio of A to B is usually written $A : B$. The quantities A and B are called the *terms* of the ratio. The first term is called the **antecedent**, the second term the **consequent**.

337. Ratios are measured by Fractions. To find what multiple or part A is of B, we divide A by B; hence the ratio $A : B$ may be measured by the fraction $\frac{A}{B}$, and we shall usually find it convenient to adopt this notation.

In order to compare two quantities, they must be expressed in terms of the same unit. Thus, the ratio of \$2 to 15 cents is measured by the fraction $\frac{2 \times 100}{15}$ or $\frac{40}{3}$.

Note. Since a ratio expresses the *number* of times that one quantity contains another, *every ratio is an abstract quantity.*

338. By Art. 136, $\qquad \frac{a}{b} = \frac{ma}{mb};$

and thus the ratio $a : b$ is equal to the ratio $ma : mb$; that is, *the value of a ratio remains unaltered if the antecedent and the consequent are multiplied or divided by the same quantity.*

339. Comparison of Ratios. Two or more ratios may be compared by reducing their equivalent fractions to a common denominator. Thus, suppose $a : b$ and $x : y$ are two ratios. Now, $\frac{a}{b} = \frac{ay}{by}$, and $\frac{x}{y} = \frac{bx}{by}$; hence the ratio $a : b$ is greater than, equal to, or less than the ratio $x : y$ according as ay is greater than, equal to, or less than bx.

340. The ratio of two fractions can be expressed as a ratio of two integers. Thus, the ratio $\frac{a}{b} : \frac{c}{d}$ is measured by $\frac{a}{b} \div \frac{c}{d}$ or $\frac{ad}{bc}$; and is therefore equivalent to the ratio $ad : bc$.

341. If either, or both, of the terms of a ratio be a surd quantity, then no two integers can be found which will *exactly* measure their ratio. Thus, the ratio $\sqrt{2} : 1$ cannot be exactly expressed by any two integers.

342. If the ratio of any two quantities can be expressed exactly by the ratio of two integers, the quantities are said to be **commensurable**; otherwise, they are said to be **incommensurable.**

Although we cannot find two integers which will exactly measure the ratio of two incommensurable quantities, we can always find two integers whose ratio differs from that required by as small a quantity as we please.

Thus, $$\frac{\sqrt{5}}{4} = \frac{2.236067 \cdots}{4} = .559016 \cdots;$$
and, therefore,
$$\frac{\sqrt{5}}{4} \text{ is} > \frac{559016}{1000000} \text{ and} < \frac{559017}{1000000};$$
and it is evident that by carrying the decimals further, any degree of approximation may be arrived at.

343. Ratios are **compounded** by multiplying together the fractions which denote them.

Ex. Find the ratio compounded of the three ratios

$$2a : 3b, \quad 6ab : 5c^2, \quad c : a.$$

The required ratio $= \dfrac{2a}{3b} \times \dfrac{6ab}{5c^2} \times \dfrac{c}{a} = \dfrac{4a}{5c}.$

344. When two identical ratios, $a : b$ and $a : b$, are compounded, the resulting ratio is $a^2 : b^2$, and is called the

duplicate ratio of $a : b$. Similarly, $a^3 : b^3$ is called the **triplicate ratio** of $a : b$. Also, $a^{\frac{1}{2}} : b^{\frac{1}{2}}$ is called the **subduplicate ratio** of $a : b$.

EXAMPLES (1) The duplicate ratio of $2a : 3b$ is $4a^2 : 9b^2$.
(2) The subduplicate ratio of $49 : 25$ is $7 : 5$.
(3) The triplicate ratio of $2x : 1$ is $8x^3 : 1$.

345. A ratio is said to be a ratio of **greater inequality**, or of **less inequality**, according as the antecedent is *greater* or *less than* the consequent.

346. If to each term of the ratio $8 : 3$ we add 4, a new ratio $12 : 7$ is obtained, and we see that it is less than the former because $\frac{12}{7}$ is clearly less than $\frac{8}{3}$.

This is a particular case of a more general proposition which we shall now prove.

A ratio of greater inequality is diminished, and a ratio of less inequality is increased, by adding the same quantity to both its terms.

Let $\frac{a}{b}$ be the ratio, and let $\frac{a+x}{b+x}$ be the new ratio formed by adding x to both its terms.

Now $$\frac{a}{b}-\frac{a+x}{b+x}=\frac{ax-bx}{b(b+x)}=\frac{x(a-b)}{b(b+x)};$$

and $a-b$ is positive or negative according as a is greater or less than b.

Hence, if a is $> b$, $\quad \frac{a}{b}$ is $> \frac{a+x}{b+x}$;

and if $\quad a$ is $< b$, $\quad \frac{a}{b}$ is $< \frac{a+x}{b+x}$,

which proves the proposition.

Similarly, it can be proved that *a ratio of greater inequality is increased, and a ratio of less inequality is diminished, by taking the same quantity from both its terms.*

347. When two or more ratios are equal, many useful propositions may be proved by introducing a single symbol to denote each of the equal ratios.

The proof of the following important theorem will illustrate the method of procedure.

If $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots,$$

each of these ratios $=\left(\dfrac{pa^n+qc^n+re^n+\cdots}{pb^n+qd^n+rf^n+\cdots}\right)^{\frac{1}{n}}$,

where p, q, r, n, are any quantities whatever.

Let $$\frac{a}{b}=\frac{c}{d}=\frac{e}{f}=\cdots=k;$$

then $$a=bk,\ c=dk,\ e=fk,\ \cdots;$$

whence $$pa^n=pb^nk^n,\ qc^n=qd^nk^n,\ re^n=rf^nk^n,\ \cdots;$$

$$\therefore\ \frac{pa^n+qc^n+re^n+\cdots}{pb^n+qd^n+rf^n+\cdots}=\frac{pb^nk^n+qd^nk^n+rf^nk^n+\cdots}{pb^n+qd^n+rf^n+\cdots}=k^n;$$

$$\therefore\ \left(\frac{pa^n+qc^n+re^n+\cdots}{pb^n+qd^n+rf^n+\cdots}\right)^{\frac{1}{n}}=k=\frac{a}{b}=\frac{c}{d}=\cdots.$$

By giving different values to p, q, r, n many particular cases of this general proposition may be deduced; or they may be proved independently by using the same method. For instance, if

$$\frac{a}{b}=\frac{c}{d}=\frac{e}{f},\ \text{each of these ratios}=\frac{a+c+e}{b+d+f};$$

a result which may be thus enunciated: *When a series of fractions are equal, each of them is equal to the sum of all the numerators divided by the sum of all the denominators.*

Ex. 1. If $\frac{x}{y}=\frac{3}{4}$ find the value of $\frac{5x-3y}{7x+2y}$.

$$\frac{5x-3y}{7x+2y}=\frac{\frac{5x}{y}-3}{\frac{7x}{y}+2}=\frac{\frac{15}{4}-3}{\frac{21}{4}+2}=\frac{3}{29}.$$

Ex. 2. Two numbers are in the ratio of $5:8$. If 9 be added to each they are in the ratio of $8:11$. Find the numbers.

Let the numbers be denoted by $5x$ and $8x$.

Then $$\frac{5x+9}{8x+9}=\frac{8}{11};\ \therefore\ x=3.$$

Hence the numbers are 15 and 24.

Ex. 3. If $A : B$ be in the duplicate ratio of $A + x : B + x$, prove that $x^2 = AB$.

By the given condition, $\left(\frac{A+x}{B+x}\right)^2 = \frac{A}{B}$;

$$\therefore\ B(A+x)^2 = A(B+x)^2,$$
$$A^2B + 2\,ABx + Bx^2 = AB^2 + 2\,ABx + Ax^2,$$
$$x^2(A - B) = AB(A - B);$$
$$\therefore\ x^2 = AB,$$

since $A - B$ is, by supposition, not zero.

EXAMPLES XXXIII. a.

Find the ratio compounded of

1. The duplicate ratio of $4 : 3$, and the ratio $27 : 8$.

2. The ratio $32 : 27$, and the triplicate ratio of $3 : 4$.

3. The subduplicate ratio of $25 : 36$, and the ratio $6 : 25$.

4. The triplicate ratio of $x : y$, and the ratio $2\,y^2 : 3\,x^2$.

5. The ratio $3\,a : 4\,b$, and the subduplicate ratio of $b^4 : a^4$.

6. If $x : y = 5 : 7$, find the value of $x + y : y - x$.

7. If $\frac{x}{y} = 3\frac{1}{3}$, find the value of $\frac{x - 3\,y}{2\,x - 5\,y}$.

8. If $b : a = 2 : 5$, find the value of $2\,a - 3\,b : 3\,b - a$.

9. If $\frac{a}{b} = \frac{3}{4}$, and $\frac{x}{y} = \frac{5}{7}$, find the value of $\frac{3\,ax - by}{4\,by - 7\,ax}$.

10. If $7\,x - 4\,y : 3\,x + y = 5 : 13$, find the ratio $x : y$.

11. If $\frac{2\,a^2 - 3\,b^2}{a^2 + b^2} = \frac{2}{41}$, find the ratio $a : b$.

12. If $2\,x : 3\,y$ be in the duplicate ratio of $2\,x - m : 3\,y - m$, prove that $m^2 = 6\,xy$.

13. If $P : Q$ be the subduplicate ratio of $P - x : Q - x$, prove that $x = \frac{PQ}{P + Q}$.

14. If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, prove that each of these ratios is equal to

$$\sqrt[3]{\frac{2\,a^2c + 3\,c^3e + 4\,e^2c}{2\,b^2d + 3\,d^3e + 4\,f^2d}}.$$

15. Two numbers are in the ratio of $3 : 4$, and if 7 be subtracted from each the remainders are in the ratio of $2 : 3$: find them.

16. What number must be taken from each term of the ratio $27 : 35$ that it may become $2 : 3$?

17. What number must be added to each term of the ratio $37:29$ that it may become $8:7$?

18. If $\frac{p}{b-c}=\frac{q}{c-a}=\frac{r}{a-b}$, show that $p+q+r=0$.

19. If $\frac{x}{b+c}=\frac{y}{c+a}=\frac{z}{a-b}$, show that $x-y+z=0$.

20. If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, show that the square root of

$$\frac{a^6b-2\,c^5e+3\,a^4c^3e^2}{b^7-2\,d^5f+3\,b^4cd^2e^2} \text{ is equal to } \frac{ace}{bdf}.$$

21. Prove that the ratio $la+mc+ne:lb+md+nf$ will be equal to each of the ratios $a:b$, $c:d$, $e:f$, if these be all equal; and that it will be intermediate in value between the greatest and least of these ratios if they be not all equal.

22. If $\frac{bx-ay}{cy-az}=\frac{cx-az}{by-ax}=\frac{z+y}{x+z}$, then will each of these fractions be equal to $\frac{x}{y}$, unless $b+c=0$.

23. If $\frac{2\,x-3\,y}{3\,z+y}=\frac{z-y}{z-x}=\frac{x+3\,z}{2\,y-3\,x}$, prove that each of these ratios is equal to $\frac{x}{y}$; hence show that either $x=y$, or $z=x+y$.

PROPORTION.

348. Definition. Four quantities are said to be in **proportion** when the ratio of the first to the second is equal to the ratio of the third to the fourth. The four quantities are called **proportionals**, or the **terms** of the proportion. Thus, if $\frac{a}{b}=\frac{c}{d}$, then a, b, c, d are proportionals. This is expressed by saying that a is to b as c is to d, and the proportion is written

$$a:b::c:d, \text{ or } a:b=c:d.$$

The terms a and d are called the *extremes*, b and c the *means*.

349. If four quantities are in proportion, the product of the extremes is equal to the product of the means.

Let a, b, c, d be the proportionals.

Then by definition $\frac{a}{b}=\frac{c}{d}$;

whence $ad = bc.$

Hence if any three terms of a proportion are given, the fourth may be found. Thus if a, c, d are given, then $b=\frac{ad}{c}$.

Conversely, if there are any four quantities, a, b, c, d, such that $ad=bc$, then a, b, c, d are proportionals; a and d being the extremes, b and c the means; or *vice versâ*.

350. Continued Proportion. Quantities are said to be in *continued proportion* when the first is to the second, as the second is to the third, as the third to the fourth; and so on. Thus a, b, c, d, $\cdots$ are in continued proportion when

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\cdots.$$

If three quantities a, b, c are in continued proportion, then

$$a:b=b:c;$$

$$\therefore\ ac=b^2. \qquad \text{(Art. 349.)}$$

In this case b is said to be a **mean proportional** between a and c; and c is said to be a **third proportional** to a and b.

351. If three quantities are proportionals, the first is to the third in the duplicate ratio of the first to the second.

Let the three quantities be a, b, c; then $\frac{a}{b}=\frac{b}{c}$.

Now $$\frac{a}{c}=\frac{a}{b}\times\frac{b}{c}=\frac{a}{b}\times\frac{a}{b}=\frac{a^2}{b^2};$$

that is, $$a:c=a^2:b^2.$$

352. The products of the corresponding terms of two or more proportions form a proportion.

If $a:b=c:d$ and $e:f=g:h$, then will

$$ae:bf=cg:dh.$$

For $\frac{a}{b}=\frac{c}{d}$ and $\frac{e}{f}=\frac{g}{h}$; $\therefore\ \frac{ae}{bf}=\frac{cg}{dh}$, or $ae:bf=cg:dh$.

COR. If $a : b = c : d,$

and $b : x = d : y,$

then $a : x = c : y.$

353. Transformations that may be made in a Proportion. If four quantities, a, b, c, d form a proportion, many other proportions may be deduced by the properties of fractions. The results of these operations are very useful, and some of them are often quoted by the annexed names borrowed from Geometry.

(1) If $a : b = c : d$, then $b : a = d : c$. **[Inversion.]**

For $\frac{a}{b} = \frac{c}{d}$; therefore $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$;

that is $$\frac{b}{a} = \frac{d}{c};$$

or $$b : a = d : c.$$

(2) If $a : b = c : d$, then $a : c = b : d$. **[Alternation.]**

For $ad = bc$; therefore $\frac{ad}{cd} = \frac{bc}{cd}$;

that is, $$\frac{a}{c} = \frac{b}{d};$$

or $$a : c = b : d.$$

(3) If $a : b = c : d$, then $a + b : b = c + d : d$ **[Composition.]**

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} + 1 = \frac{c}{d} + 1$;

that is, $$\frac{a+b}{b} = \frac{c+d}{d};$$

or $$a + b : b = c + d : d.$$

(4) If $a : b = c : d$, then $a - b : b = c - d : d$. **[Division.]**

For $\frac{a}{b} = \frac{c}{d}$; therefore $\frac{a}{b} - 1 = \frac{c}{d} - 1$;

that is, $$\frac{a-b}{b} = \frac{c-d}{d};$$

or $$a - b : b = c - d : d.$$

(5) If $a:b=c:d$, then $a+b:a-b=c+d:c-d$.

[Composition and Division.]

For by (3) $\dfrac{a+b}{b}=\dfrac{c+d}{d}$; and by (4) $\dfrac{a-b}{b}=\dfrac{c-d}{d}$;

$\therefore$ by division, $$\frac{a+b}{a-b}=\frac{c+d}{c-d};$$

or $$a+b:a-b=c+d:c-d.$$

Several other proportions may be proved in a similar way.

Ex. 1. If $$a:b=c:d=e:f,$$

show that $$2a^2+3c^2-5e^2:2b^2+3d^2-5f^2=ae:bf.$$

Let $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=k$; then $a=bk$, $c=dk$, $e=fk$;

$$\therefore \frac{2a^2+3c^2-5e^2}{2b^2+3d^2-5f^2}=\frac{2b^2k^2+3d^2k^2-5f^2k^2}{2b^2+3d^2-5f^2}=k^2=\frac{a}{b}\times\frac{e}{f}=\frac{ae}{bf},$$

or $$2a^2+3c^2-5e^2:2b^2+3d^2-5f^2=ae:bf.$$

Ex. 2. If

$$(3a+6b+c+2d)(3a-6b-c+2d)$$
$$=(3a-6b+c-2d)(3a+6b-c-2d),$$

prove that a, b, c, d are in proportion.

We have $$\frac{3a+6b+c+2d}{3a-6b+c-2d}=\frac{3a+6b-c-2d}{3a-6b-c+2d}.$$ [Art. 349.]

Composition and division, $$\frac{2(3a+c)}{2(6b+2d)}=\frac{2(3a-c)}{2(6b-2d)}.$$

Alternation, $$\frac{3a+c}{3a-c}=\frac{6b+2d}{6b-2d}.$$

Again, composition and division, $\dfrac{6a}{2c}=\dfrac{12b}{4d}$;

whence $$a:b=c:d.$$

Ex. 3. Solve the equation $\dfrac{x^2+x-2}{x-2}=\dfrac{4x^2+5x-6}{5x-6}$.

Division, $$\frac{x^2}{x-2}=\frac{4x^2}{5x-6};$$

whence, dividing by x^2, which gives a solution $x=0$, [Art. 291, note.]

$$\frac{1}{x-2}=\frac{4}{5x-6};\text{ whence, } x=-2;$$

and therefore the roots are 0, -2.

EXAMPLES XXXIII. b.

Find a fourth proportional to

1. a, ab, c. **2.** $a^2, 2ab, 3b^2$. **3.** $x^3, xy, 5x^2y$.

Find a third proportional to

4. a^2b, ab. **5.** $x^3, 2x^2$. **6.** $3x, 6xy$. **7.** $1, x$.

Find a mean proportional between

8. a^2, b^2. **9.** $2x^3, 8x$. **10.** $12ax^2, 3a^3$. **11.** $27a^2b^3, 3b$.

If a, b, c be three proportionals, show that

12. $a : a + b = a - b : a - c$.

13. $(b^2 + bc + c^2)(ac - bc + c^2) = b^4 + ac^3 + c^4$.

If $a : b = c : d$, prove that

14. $ab + cd : ab - cd = a^2 + c^2 : a^2 - c^2$.

15. $a^2 + ac + c^2 : a^2 - ac + c^2 = b^2 + bd + d^2 : b^2 - bd + d^2$.

16. $a : b = \sqrt{3a^2 + 5c^2} : \sqrt{3b^2 + 5d^2}$.

17. $\frac{a}{p} + \frac{b}{q} : a = \frac{c}{p} + \frac{d}{q} : c$.

18. $\frac{b}{a} + \frac{a}{b} : \frac{ab}{a^2 + b^2} = \frac{d}{c} + \frac{c}{d} : \frac{cd}{c^2 + d^2}$.

Solve the equations:

19. $3x - 1 : 6x - 7 = 7x - 10 : 9x + 10$.

20. $x - 12 : y + 3 = 2x - 19 : 5y - 13 = 5 : 14$.

21. $\frac{x^2 - 2x + 3}{2x - 3} = \frac{x^2 - 3x + 5}{3x - 5}$. **22.** $\frac{2x - 1}{x^2 + 2x - 1} = \frac{x + 4}{x^2 + x + 4}$.

23. If $(a + b - 3c - 3d)(2a - 2b - c + d)$
$$= (2a + 2b - c - d)(a - b - 3c + 3d)$$
prove that a, b, c, d are proportionals.

24. If a, b, c, d are in continued proportion, prove that
$$a : d = a^3 + b^3 + c^3 : b^3 + c^3 + d^3.$$

25. If b is a mean proportional between a and c, show that
$4a^2 - 9b^2$ is to $4b^2 - 9c^2$ in the duplicate ratio of a to b.

26. If a, b, c, d are in continued proportion, prove that $b + c$ is a mean proportional between $a + b$ and $c + d$.

27. If $$a + b : b + c = c + d : d + a,$$
prove that $a = c$, or $a + b + c + d = 0$.

VARIATION.

354. Definition. One quantity A is said to **vary directly** as another B, when the two quantities so depend upon each other that if B is changed, A is changed *in the same ratio.*

Note. The word *directly* is often omitted, and A is said to vary as B.

355. For instance: if a train moving at a uniform rate travels 40 miles in 60 minutes, it will travel 20 miles in 30 minutes, 80 miles in 120 minutes, and so on; the distance in each case being increased or diminished in the same ratio as the time. This is expressed by saying that when the velocity is uniform *the distance is proportional to the time*, or more briefly, *the distance varies as the time.*

356. The Symbol of Variation. The symbol $\propto$ is used to denote variation; so that $A \propto B$ is read "A varies as B."

357. If A varies as B, then A is equal to B multiplied by some constant quantity.

For suppose that $a_1, a_2, a_3 \ldots, b_1, b_2, b_3 \ldots$ are corresponding values of A and B.

Then, by definition, $\frac{A}{a_1}=\frac{B}{b_1}$; $\frac{A}{a_2}=\frac{B}{b_2}$; $\frac{A}{a_3}=\frac{B}{b_3}$; and so on;

$$\therefore \frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}=\ldots, \text{ each being equal to } \frac{A}{B}.$$

Hence $\dfrac{\text{any value of } A}{\text{the corresponding value of } B}$ is always the same;

that is, $\frac{A}{B}=m$, where m is constant.

$$\therefore A=mB.$$

358. Definition. One quantity A is said to **vary inversely** as another B when A varies *directly* as the reciprocal of B. [See Art. 176.]

Thus if A varies inversely as B, $A=\frac{m}{B}$, where m is constant.

The following is an illustration of inverse variation: If 6 men do a certain work in 8 hours, 12 men would do the same work in 4 hours, 2 men in 24 hours; and so on. Thus it appears that when the number of men is increased the time is proportionately decreased; and *vice versâ*.

359. Definition. One quantity is said to **vary jointly** as a number of others when it varies directly as their product.

Thus A varies jointly as B and C when $A=mBC$. For instance, the interest on a sum of money varies jointly as the principal, the time, and the rate per cent.

360. Definition. A is said to **vary directly** as B and **inversely** as C when A varies as $\frac{B}{C}$.

361. Grouping the principles of Arts. 357–360, we have

$A=mB$, if A varies directly as B,

$A=\frac{m}{B}$, if A varies inversely as B.

$A=mBC$, if A varies jointly as B and C,

$A=\frac{mB}{C}$, if A varies directly as B and inversely as C.

362. If A varies as B when C is constant, and A varies as C when B is constant, then will A vary as BC when both B and C vary.

The variation of A depends partly on that of B and partly on that of C. Suppose these latter variations to take place separately, each in its turn producing its own effect on A; also let a, b, c be certain simultaneous values of A, B, C.

1. *Let C be constant* while B changes to b; then A must undergo a partial change and will assume some intermediate value a', where

$$\frac{A}{a'}=\frac{B}{b}. \qquad \qquad (1).$$

2. *Let B be constant,* that is, let it retain its value b, while C changes to c; then A must complete its change and pass from its intermediate value a' to its final value a, where

$$\frac{a'}{a} = \frac{C}{c} \quad \ldots\ldots\ldots \quad (2).$$

From (1) and (2) $$\frac{A}{a'} \times \frac{a'}{a} = \frac{B}{b} \times \frac{C}{c};$$

that is, $$A = \frac{a}{bc} \cdot BC,$$

or $$A \text{ varies as } BC.$$

363. The following are illustrations of the theorem proved in the last article.

The amount of work done by *a given number of men* varies directly as the number of days they work, and the amount of work done *in a given time* varies directly as the number of men; therefore when the number of days and the number of men are both variable, the amount of work will vary as the product of the number of men and the number of days.

Again, in Geometry the area of a triangle varies directly as its base when the height is constant, and directly as the height when the base is constant; and when both the height and base are variable, the area varies as the product of the numbers representing the height and the base.

Ex. 1. If $A \propto B$, and $C \propto D$, then will $AC \propto BD$.

For, by supposition, $A = mB$, $C = nD$, where m and n are constants.

Therefore $AC = mnBD$; and as mn is constant, $AC \propto BD$.

Ex. 2. If x varies inversely as $y^2 - 1$, and is equal to 24 when $y = 10$; find x when $y = 5$.

By supposition, $x = \dfrac{m}{y^2 - 1}$, where m is constant.

Putting $x = 24$, $y = 10$, we obtain $24 = \dfrac{m}{99}$,

whence $$m = 24 \times 99;$$

$$\therefore \; x = \frac{24 \times 99}{y^2 - 1};$$

hence, putting $y = 5$, we obtain $x = 99$.

Ex. 3. The volume of a pyramid varies jointly as its height and the area of its base; and when the area of the base is 60 square feet and the height 14 feet, the volume is 280 cubic feet. What is the area of the base of a pyramid whose volume is 390 cubic feet and whose height is 26 feet?

Let V denote the volume, A the area of the base, and h the height; then $V = mAh$, where m is constant.

Substituting the given values of V, A, h, we have

$$280 = m \times 60 \times 14;$$

$$\therefore\ m = \frac{280}{60 \times 14} = \frac{1}{3}.$$

$$\therefore\ V = \tfrac{1}{3} Ah.$$

Also when $V = 390,\ h = 26;$

$$\therefore\ 390 = \tfrac{1}{3} A \times 26;$$

$$\therefore\ A = 45.$$

Hence the area of the base is 45 square feet.

EXAMPLES XXXIII. c.

1. If $x \propto y$, and $y = 7$ when $x = 18$, find x when $y = 21$.

2. If $x \propto y$, and $y = 3$ when $x = 2$, find y when $x = 18$.

3. A varies jointly as B and C; and $A = 6$ when $B = 3$, $C = 2$: find A when $B = 5$, $C = 7$.

4. A varies jointly as B and C; and $A = 9$ when $B = 5$, $C = 7$: find B when $A = 54$, $C = 10$.

5. If $x \propto \frac{1}{y}$, and $y = 4$ when $x = 15$, find y when $x = 6$.

6. If $y \propto \frac{1}{x}$, and $y = 1$ when $x = 1$, find x when $y = 5$.

7. A varies as B directly, and as C inversely; and $A = 10$ when $B = 15$, $C = 6$: find A when $B = 8$, $C = 2$.

8. If x varies as y directly, and as z inversely, and $x = 14$ when $y = 10$, $z = 14$, find z when $x = 49$, $y = 45$.

9. If $x \propto \frac{1}{y}$, and $y \propto \frac{1}{z}$, prove that $z \propto x$.

10. If $a \propto b$, prove that $a^n \propto b^n$.

11. If $x \propto z$ and $y \propto z$, prove that $x^2 - y^2 \propto z^2$.

12. If $3a + 7b \propto 3a + 13b$, and when $a = 5$, $b = 3$, find the equation between a and b.

13. If $5x - y \propto 10x - 11y$, and when $x = 7$, $y = 5$, find the equation between x and y.

14. If the cube of x varies as the square of y, and if $x = 3$ when $y = 5$, find the equation between x and y.

15. If the square root of a varies as the cube root of b, and if $a = 4$ when $b = 8$, find the equation between a and b.

16. If y varies inversely as the square of x, and if $y = 8$ when $x = 3$, find x when $y = 2$.

17. If $x \propto y + a$, where a is constant, and $x = 15$ when $y = 1$, and $x = 35$ when $y = 5$; find x when $y = 2$.

18. If $a + b \propto a - b$, prove that $a^2 + b^2 \propto ab$; and if $a \propto b$, prove that $a^2 - b^2 \propto ab$.

19. If y be the sum of three quantities which vary as x, x^2, x^3 respectively, and when $x = 1$, $y = 4$, when $x = 2$, $y = 8$, and when $x = 3$, $y = 18$, express y in terms of x.

20. Given that the area of a circle varies as the square of its radius, and that the area of a circle is 154 square feet when the radius is 7 feet: find the area of a circle whose radius is 10 feet 6 inches.

21. The area of a circle varies as the square of its diameter: prove that the area of a circle whose diameter is $2\frac{1}{2}$ inches is equal to the sum of the areas of two circles whose diameters are $1\frac{1}{2}$ and 2 inches respectively.

22. The pressure of wind on a plane surface varies jointly as the area of the surface, and the square of the wind's velocity. The pressure on a square foot is 1 pound when the wind is moving at the rate of 15 miles per hour: find the velocity of the wind when the pressure on a square yard is 16 pounds.

23. The value of a silver coin varies directly as the square of its diameter, while its thickness remains the same; it also varies directly as its thickness while its diameter remains the same. Two silver coins have their diameters in the ratio of 4 : 3. Find the ratio of their thicknesses if the value of the first be four times that of the second.

24. The volume of a circular cylinder varies as the square of the radius of the base when the height is the same, and as the height when the base is the same. The volume is 88 cubic feet when the height is 7 feet, and the radius of the base is 2 feet: what will be the height of a cylinder on a base of radius 9 feet, when the volume is 396 cubic feet?

CHAPTER XXXIV.

ARITHMETICAL, GEOMETRICAL, AND HARMONICAL PROGRESSIONS.

364. A succession of quantities formed according to some fixed law is called a **series.** The separate quantities are called **terms** of the series.

ARITHMETICAL PROGRESSION.

365. DEFINITION. Quantities are said to be in **Arithmetical Progression** when they increase or decrease by a *common difference.*

Thus each of the following series forms an Arithmetical Progression:

$$3,\ 7,\ 11,\ 15,\ \cdots$$

$$8,\ 2,\ -4,\ -10,\ \cdots$$

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots.$$

The common difference is found by subtracting *any* term of the series from that which *follows* it. In the first of the above examples the common difference is 4; in the second it is -6; in the third it is d.

366. The Last, or *n*th Term, of an A. P. If we examine the series

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots$$

we notice that in any term the coefficient of d is always less by one than the number of the term in the series.

Thus the 3d term is $a+2d$;

6th term is $a+5d$;

20th term is $a+19d$;

and, generally, the pth term is $a+(p-1)d$.

If n be the number of terms, and if l denote the last, or nth term, we have

$$l = a + (n-1)d.$$

367. The Sum of *n* Terms in A. P. Let a denote the first term, d the common difference, and n the number of terms. Also let l denote the last term, and S the required sum; then

$$S = a + (a+d) + (a+2d) + \cdots + (l-2d) + (l-d) + l.$$

and, by writing the series in the reverse order,

$$S = l + (l-d) + (l-2d) + \cdots + (a+2d) + (a+d) + a.$$

Adding together these two series,

$$2S = (a+l) + (a+l) + (a+l) + \cdots \text{ to } n \text{ terms} = n(a+l),$$

$$\therefore\ S = \frac{n}{2}(a+l) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Since $$l = a + (n-1)d \quad . \quad . \quad . \quad . \quad . \quad (2);$$

$$\therefore\ S = \frac{n}{2}\{2a + (n-1)d\} \quad . \quad . \quad . \quad (3).$$

368. In the last article we have three useful formulæ (1), (2), (3); in each of these any one of the letters may denote the unknown quantity when the three others are known.

Ex. 1. Find the 20th and 35th terms of the series

$$38,\ 36,\ 34,\ \cdots.$$

Here the common difference is $36-38$, or -2.

$$\therefore \text{ the 20th term} = 38 + 19(-2) = 0;$$

and the 35th term $= 38 + 34(-2) = -30$.

Ex. 2. Find the sum of the series $5\frac{1}{2}$, $6\frac{3}{4}$, 8, $\cdots$ to 17 terms.

Here the common difference is $1\frac{1}{4}$; hence from (3)

The sum $= \frac{17}{2}\{2 \times \frac{11}{2} + 16 \times 1\frac{1}{4}\}$

$= \frac{17}{2}(11+20) = \dfrac{17 \times 31}{2} = 263\frac{1}{2}.$

Ex. 3. The first term of a series is 5, the last 45, and the sum 400: find the number of terms, and the common difference.

If n be the number of terms, then from (1),

$$400 = \frac{n}{2}(5 + 45);$$

whence $$n = 16.$$

If d be the common difference,

$$45 = \text{the 16th term} = 5 + 15d;$$

whence $$d = 2\tfrac{2}{3}.$$

EXAMPLES XXXIV. a.

1. Find the 27th and 41st terms in the series 5, 11, 17, ⋯.
2. Find the 13th and 109th terms in the series 71, 70, 69, ⋯.
3. Find the 17th and 54th terms in the series 10, $11\frac{1}{2}$, 13, ⋯.
4. Find the 20th and 13th terms in the series -3, -2, -1, ⋯.
5. Find the 90th and 16th terms in the series -4, 2.5, 9, ⋯.
6. Find the 37th and 89th terms in the series -2.8, 0, 2.8, ⋯.

Find the last term in the following series:

7. 5, 7, 9, ⋯ to 20 terms.
8. 7, 3, -1, ⋯ to 15 terms.
9. $13\frac{1}{2}$, 9, $4\frac{1}{2}$, ⋯ to 13 terms.
10. .6, 1.2, 1.8, ⋯ to 12 terms.
11. 2.7, 3.4, 4.1, ⋯ to 11 terms.
12. x, $2x$, $3x$, ⋯ to 25 terms.
13. $a - d$, $a + d$, $a + 3d$, ⋯ to 30 terms.
14. $2a - b$, $4a - 3b$, $6a - 5b$, ⋯ to 40 terms.

Find the last term and sum of the following series:

15. 14, 64, 114, ⋯ to 20 terms.
16. 1, 1.2, 1.4, ⋯ to 12 terms.
17. 9, 5, 1, ⋯ to 100 terms.
18. $\frac{1}{4}$, $-\frac{1}{4}$, $-\frac{3}{4}$, ⋯ to 21 terms.
19. $3\frac{1}{2}$, 1, $-1\frac{1}{2}$, ⋯ to 19 terms.
20. 64, 96, 128, ⋯ to 16 terms.

Find the sum of the following series:

21. 5, 9, 13, ⋯ to 19 terms.
22. 12, 9, 6, ⋯ to 23 terms.
23. 4, $5\frac{1}{4}$, $6\frac{1}{2}$, ⋯ to 37 terms.
24. $10\frac{1}{2}$, 9, $7\frac{1}{2}$, ⋯ to 94 terms.
25. -3, 1, 5, ⋯ to 17 terms.
26. 10, $9\frac{2}{3}$, $9\frac{1}{3}$, ⋯ to 21 terms.
27. p, $3p$, $5p$, ⋯ to p terms.
28. $3a$, a, $-a$, ⋯ to a terms.
29. a, 0, $-a$, ⋯ to a terms.
30. $-3q$, $-q$, q, ⋯ to p terms.

Find the number of terms and the common difference when

31. The first term is 3, the last term 90, and the sum 1395.
32. The first term is 79, the last term 7, and the sum 1075.
33. The sum is 24, the first term 9, the last term -6.
34. The sum is 714, the first term 1, the last term $58\frac{1}{2}$.
35. The last term is -16, the sum -133, the first term -3.
36. The first term is -75, the sum -740, the last term 1.
37. The first term is a, the last $13a$, and the sum $49a$.
38. The sum is $-320x$, the first term $3x$, the last term $-35x$.

369. If *any two* terms of an Arithmetical Progression be given, the series can be completely determined; for the data furnish *two* simultaneous equations, the solution of which will give the first term and the common difference.

Ex. Find the series whose 7th and 51st terms are -3 and -355 respectively.

If a be the first term, and d the common difference,

$$-3 = \text{the 7th term} = a + 6d;$$

and $$-355 = \text{the 51st term} = a + 50d;$$

whence, by subtraction, $$-352 = 44d;$$

$\therefore\ d = -8$; and, consequently, $a = 45$.

Hence the series is 45, 37, 29 ....

370. Arithmetic Mean. When three quantities are in Arithmetical Progression, the middle one is said to be the *arithmetic mean* of the other two.

Thus a is the arithmetic mean between $a - d$ and $a + d$.

371. To find the arithmetic mean between two given quantities.

Let a and b be the two quantities; A the arithmetic mean. Then, since a, A, b, are in A.P., we must have

$$b - A = A - a,$$

each being equal to the common difference;

whence $$A = \frac{a + b}{2}.$$

372. Between two given quantities it is always possible to insert any number of terms such that the whole series thus formed shall be in A. P.; and by an extension of the definition in Art. 370, the terms thus inserted are called the *arithmetic means*.

Ex. Insert 20 arithmetic means between 4 and 67.

Including the extremes the number of terms will be 22; so that we have to find a series of 22 terms in A. P., of which 4 is the first and 67 the last.

Let d be the common difference;

then $\qquad 67 = \text{the 22d term}, \; = 4 + 21d;$

whence $d = 3$, and the series is 4, 7, 10, ⋯ 61, 64, 67;

and the required means are 7, 10, 13, ⋯ 58, 61, 64.

373. To insert a given number of arithmetic means between two given quantities.

Let a and b be the given quantities, m the number of means.

Including the extremes the number of terms will be $m+2$; so that we have to find a series of $m+2$ terms in A. P., of which a is the first, and b is the last.

Let d be the common difference;

then
$$b = \text{the } (m+2)\text{th term}$$
$$= a + (m+1)d;$$

whence
$$d = \frac{b-a}{m+1};$$

and the required *means* are

$$a + \frac{b-a}{m+1},\; a + \frac{2(b-a)}{m+1}, \cdots a + \frac{m(b-a)}{m+1}$$

Ex. 1. Find the 30th term of an A. P. of which the first term is 17, and the 100th term -16.

Let d be the common difference;

then
$$-16 = \text{the 100th term}$$
$$= 17 + 99d;$$
$$\therefore d = -\tfrac{1}{3}.$$

The 30th term $\qquad = 17 + 29(-\frac{1}{3}) = 7\frac{1}{3}.$

Ex. 2. The sum of three numbers in A. P. is 33, and their product is 792: find them.

Let a be the *middle* number, d the common difference; then the three numbers are $a - d$, a, $a + d$.

Hence $$a - d + a + a + d = 33;$$
whence $a = 11$; and the three numbers are $11 - d$, 11, $11 + d$.

$$\therefore\ 11(11 + d)(11 - d) = 792,$$
$$121 - d^2 = 72,$$
$$d = \pm 7;$$
and the numbers are 4, 11, 18.

Ex. 3. How many terms of the series 24, 20, 16, ··· must be taken that the sum may be 72?

Let the number of terms be n; then, since the common difference is $20 - 24$, or -4, we have from (3), Art. 367,

$$72 = \frac{n}{2}\{2 \times 24 + (n - 1)(-4)\}$$
$$= 24n - 2n(n - 1);$$
whence $$n^2 - 13n + 36 = 0,$$
or $$(n - 4)(n - 9) = 0;$$
$$\therefore\ n = 4 \text{ or } 9.$$

Both of these values satisfy the conditions of the question; for if we write down the first 9 terms, we get 24, 20, 16, 12, 8, 4, 0, -4, -8; and, as the last five terms destroy each other, the sum of 9 terms is the same as that of 4 terms.

Ex. 4. An A. P. consists of 21 terms; the sum of the three terms in the middle is 129, and of the last three is 237; find the series.

Let a be the first term, and d the common difference. Then

$$237 = \text{the sum of the last three terms}$$
$$= a + 20d + a + 19d + a + 18d = 3a + 57d;$$
whence $$a + 19d = 79 \quad \ldots\ldots\ldots \quad (1).$$

Again, the three middle terms are the 10th, 11th, 12th;

hence $$129 = \text{the sum of the three middle terms}$$
$$= a + 9d + a + 10d + a + 11d = 3a + 30d;$$
whence $$a + 10d = 43 \quad \ldots\ldots\ldots \quad (2).$$

From (1) and (2), we obtain $d = 4$, $a = 3$.

Hence the series is 3, 7, 11, ··· 83.

EXAMPLES XXXIV. b.

Find the series in which

1. The 27th term is 186, and the 45th term 312.

2. The 5th term is 1, and the 31st term -77.

3. The 15th term is -25, and the 23rd term -41.

4. The 9th term is -11, and the 102nd term $-150\frac{1}{2}$.

5. The 15th term is 25, and the 29th term 46.

6. The 16th term is 214, and the 51st term 739.

7. The 3rd and 7th terms of an A. P. are 7 and 19; find the 15th term.

8. The 54th and 4th terms are -125 and 0; find the 42nd term.

9. The 31st and 2nd terms are $\frac{1}{2}$ and $7\frac{3}{4}$; find the 59th term.

10. Insert 15 arithmetic means between 71 and 23.

11. Insert 17 arithmetic means between 93 and 69.

12. Insert 14 arithmetic means between $-7\frac{1}{5}$ and $-2\frac{1}{5}$.

13. Insert 16 arithmetic means between 7.2 and -6.4.

14. Insert 36 arithmetic means between $8\frac{1}{2}$ and $2\frac{1}{3}$.

How many terms must be taken of

15. The series 42, 39, 36, ... to make 315?

16. The series -16, -15, -14, ... to make -100?

17. The series $15\frac{2}{3}$, $15\frac{1}{3}$, 15, ... to make 120?

18. The series 20, $18\frac{3}{4}$, $17\frac{1}{2}$, ... to make $162\frac{1}{2}$?

19. The series $-10\frac{1}{2}$, -9, $-7\frac{1}{2}$, ... to make -42?

20. The series $-6\frac{4}{5}$, $-6\frac{2}{5}$, -6, ... to make $-52\frac{4}{5}$?

21. The sum of three numbers in A. P. is 39, and their product is 2184: find them.

22. The sum of three numbers in A. P. is 12, and the sum of their squares is 66: find them.

23. The sum of five numbers in A. P. is 75, and the product of the greatest and least is 161: find them.

24. The sum of five numbers in A. P. is 40, and the sum of their squares is 410: find them.

25. The 12th, 85th, and last terms of an A. P. are 38, 257, 395 respectively: find the number of terms.

GEOMETRICAL PROGRESSION.

374. Definition. Quantities are said to be in **Geometrical Progression** when they increase or decrease by a *constant factor*.

Thus each of the following series forms a Geometrical Progression:

$$3,\ 6,\ 12,\ 24,\ \ldots$$

$$1,\ -\tfrac{1}{3},\ \tfrac{1}{9},\ -\tfrac{1}{27},\ \ldots$$

$$a,\ ar,\ ar^2,\ ar^3,\ \ldots$$

The constant factor is also called the *common ratio*, and it is found by dividing *any* term by that which immediately *precedes* it. In the first of the above examples the common ratio is 2; in the second it is $-\frac{1}{3}$; in the third it is r.

375. The Last, or *n*th Term, of a G. P. If we examine the series

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \ldots$$

we notice that *in any term the index of r is always less by one than the number of the term in the series.*

Thus the 3rd term is ar^2;

the 6th term is ar^5;

the 20th term is ar^{19};

and, generally, the pth term is ar^{p-1}.

If n be the number of terms, and if l denote the last, or nth term, we have $l = ar^{n-1}$.

Ex. Find the 8th term of the series $-\frac{1}{3},\ \frac{1}{2},\ -\frac{3}{4},\ \ldots$

The common ratio is $\frac{1}{2} \div (-\frac{1}{3})$, or $-\frac{3}{2}$;

$$\therefore \text{ the 8th term} = -\tfrac{1}{3} \times (-\tfrac{3}{2})^7$$

$$= -\tfrac{1}{3} \times -\tfrac{2187}{128} = \tfrac{729}{128}.$$

376. Geometric Mean. When three quantities are in Geometrical Progression the middle one is called the *geometric mean* between the other two.

377. To find the geometric mean between two given quantities.

Let a and b be the two quantities; G the geometric mean. Then since a, G, b are in G. P.,

$$\frac{b}{G}=\frac{G}{a},$$

each being equal to the common ratio;

$$\therefore\ G^2=ab;$$

whence $$G=\sqrt{ab}.$$

378. To insert a given number of geometric means between two given quantities.

Let a and b be the given quantities, m the number of means.

There will be $m+2$ terms; so that we have to find a series of $m+2$ terms in G. P., of which a is the first and b the last.

Let r be the common ratio;

then $b=$ the $(m+2)$th term $=ar^{m+1}$;

$$\therefore\ r^{m+1}=\frac{b}{a};$$

$$\therefore\ r=\left(\frac{b}{a}\right)^{\frac{1}{m+1}} \quad \ldots\ldots \quad (1).$$

Hence the required means are ar, ar^2, $\cdots ar^m$, where r has the value found in (1).

Ex. Insert 4 geometric means between 160 and 5.

We have to find 6 terms in G. P. of which 160 is the first, and 5 the sixth.

Let r be the common ratio;

then $5=$ the sixth term $=160\,r^5$;

$$\therefore\ r^5=\tfrac{1}{32};$$

whence, *by trial*, $r=\frac{1}{2}$;

and the means are 80, 40, 20, 10.

379. The Sum of n Terms in G. P. Let a be the first term, r the common ratio, n the number of terms, and S the sum required. Then

$$S = a + ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1};$$

multiplying every term by r, we have

$$rS = ar + ar^2 + \cdots + ar^{n-2} + ar^{n-1} + ar^n.$$

Hence by subtraction,

$$rS - S = ar^n - a;$$

$$\therefore\ (r-1)S = a(r^n - 1);$$

$$\therefore\ S = \frac{a(r^n-1)}{r-1} \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Changing the signs in numerator and denominator

$$S = \frac{a(1-r^n)}{1-r} \quad . \quad . \quad . \quad . \quad . \quad (2).$$

NOTE. It will be found convenient to remember both forms given above for S, using (2) in all cases except when r is *positive and greater than* 1.

Since $ar^{n-1} = l$, it follows that $ar^n = rl$, and formula (1) may be written

$$S = \frac{rl - a}{r-1}.$$

Ex. 1. Sum the series 81, 54, 36, $\cdots$ to 9 terms.

The common ratio $= \frac{54}{81} = \frac{2}{3}$, which is less than 1;

hence the sum $= \dfrac{81\{1-(\frac{2}{3})^9\}}{1-\frac{2}{3}} = 243\{1-(\frac{2}{3})^9\}$

$= 243 - \frac{512}{81} = 236\frac{55}{81}$.

Ex. 2. Sum the series $\frac{2}{3}$, -1, $\frac{3}{2}$, $\cdots$ to 7 terms.

The common ratio $= -\frac{3}{2}$; hence by formula (2)

the sum $= \dfrac{\frac{2}{3}\{1-(-\frac{3}{2})^7\}}{1+\frac{3}{2}} = \dfrac{\frac{2}{3}\{1+\frac{2187}{128}\}}{\frac{5}{2}}$

$= \frac{2}{3} \times \frac{2315}{128} \times \frac{2}{5} = \frac{463}{96}$.

EXAMPLES XXXIV. c.

1. Find the 5th and 8th terms of the series 3, 6, 12, $\cdots$.
2. Find the 10th and 16th terms of the series 256, 128, 64, $\cdots$.
3. Find the 7th and 11th terms of the series 64, -32, 16, $\cdots$.

4. Find the 8th and 12th terms of the series 81, -27, 9, $\cdots$.

5. Find the 14th and 7th terms of the series $\frac{1}{64}$, $\frac{1}{32}$, $\frac{1}{16}$, $\cdots$.

6. Find the 4th and 8th terms of the series .008, .04, .2, $\cdots$.

Find the last term in the following series:

7. 2, 4, 8, $\cdots$ to 9 terms.

8. 2, -6, 18, $\cdots$ to 8 terms.

9. 2, 3, $4\frac{1}{2}$, $\cdots$ to 6 terms.

10. 3, -3^2, 3^3, $\cdots$ to $2n$ terms.

11. x, x^3, x^5, $\cdots$ to p terms.

12. x, 1, $\frac{1}{x}$, $\cdots$ to 30 terms.

13. Insert 3 geometric means between 486 and 6.

14. Insert 4 geometric means between $\frac{1}{8}$ and 128.

15. Insert 6 geometric means between 56 and $-\frac{7}{16}$.

16. Insert 5 geometric means between $\frac{32}{81}$ and $4\frac{1}{2}$.

Find the last term and the sum of the following series:

17. 3, 6, 12, $\cdots$ to 8 terms.

18. 6, -18, 54, $\cdots$ to 6 terms.

19. 64, 32, 16, $\cdots$ to 10 terms.

20. 8.1, 2.7, .9, $\cdots$ to 7 terms.

21. $\frac{1}{72}$, $\frac{1}{24}$, $\frac{1}{8}$, $\cdots$ to 8 terms.

22. $4\frac{1}{2}$, $1\frac{1}{2}$, $\frac{1}{2}$, $\cdots$ to 9 terms.

Find the sum of the series:

23. 3, -1, $\frac{1}{3}$, $\cdots$ to 6 terms.

24. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{9}$, $\cdots$ to 7 terms.

25. $-\frac{2}{5}$, $\frac{1}{2}$, $-\frac{5}{8}$, $\cdots$ to 6 terms.

26. 1, $-\frac{1}{2}$, $\frac{1}{4}$, $\cdots$ to 12 terms.

27. 9, -6, 4, $\cdots$ to 7 terms.

28. $\frac{2}{3}$, $-\frac{1}{6}$, $\frac{1}{24}$, $\cdots$ to 8 terms.

29. 1, 3, 3^2, $\cdots$ to p terms.

30. 2, -4, 8, $\cdots$ to $2p$ terms.

31. $\frac{1}{\sqrt{3}}$, 1, $\frac{3}{\sqrt{3}}$, $\cdots$ to 8 terms.

32. $\sqrt{a}$, $\sqrt{a^3}$, $\sqrt{a^5}$, $\cdots$ to a terms.

33. $\frac{1}{\sqrt{2}}$, -2, $\frac{8}{\sqrt{2}}$, $\cdots$ to 7 terms.

34. $\sqrt{2}$, $\sqrt{6}$, $3\sqrt{2}$, $\cdots$ to 12 terms.

380. Infinite Geometrical Series. Consider the series

$$1, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \cdots.$$

$$\text{The sum to } n \text{ terms } = \frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} = 2\left(1-\frac{1}{2^n}\right) = 2-\frac{1}{2^{n-1}}.$$

From this result it appears that however many terms be taken the sum of the above series is always less than 2. Also we see that, by making n sufficiently large, we can make the fraction $\frac{1}{2^{n-1}}$ as small as we please. Thus by taking a sufficient number of terms the sum can be made to differ by as little as we please from 2.

In the next article a more general case is discussed.

381. Sum to Infinity. From Art. 379 we have

$$S = \frac{a(1-r^n)}{1-r} = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

Suppose r is a proper fraction; then the greater the value of n the smaller is the value of r^n, and consequently of $\frac{ar^n}{1-r}$; and therefore by making n sufficiently large, we can make the sum of n terms of the series differ from $\frac{a}{1-r}$ by as small a quantity as we please.

This result is usually stated thus: *the sum of an infinite number of terms of a decreasing Geometrical Progression is* $\frac{a}{1-r}$; or more briefly, *the sum to infinity is* $\frac{a}{1-r}$.

382. Recurring decimals furnish a good illustration of Infinite Geometrical Progressions.

Ex. Find the value of $.4\dot{2}\dot{3}$.

$$.4\dot{2}\dot{3} = .4232323\cdots = \frac{4}{10} + \frac{23}{1000} + \frac{23}{100000} + \cdots = \frac{4}{10} + \frac{23}{10^3} + \frac{23}{10^5} + \cdots$$

$$= \frac{4}{10} + \frac{23}{10^3}\left(1 + \frac{1}{10^2} + \frac{1}{10^4} + \cdots\right) = \frac{4}{10} + \frac{23}{10^3} \cdot \frac{1}{1 - \frac{1}{10^2}}$$

$$= \frac{4}{10} + \frac{23}{10^3} \cdot \frac{100}{99} = \frac{4}{10} + \frac{23}{990} = \frac{419}{990},$$

which agrees with the value found by the usual arithmetical rule.

EXAMPLES XXXIV. d.

Sum to infinity the following series:

1. 9, 6, 4, $\cdots$ **3.** $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ **5.** $\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \cdots$ **7.** .9, .03, .001, $\cdots$

2. 12, 6, 3, $\cdots$ **4.** $\frac{1}{2}, -\frac{1}{4}, \frac{1}{8}, \cdots$ **6.** $\frac{8}{5}, -1, \frac{5}{8}, \cdots$ **8.** .8, $-$.4, .2, $\cdots$

Find by the method of Art. 382, the value of

9. $.\dot{3}$. **10.** $.1\dot{6}$. **11.** $.\dot{2}\dot{4}$. **12.** $.\dot{3}7\dot{8}$. **13.** $.\dot{0}3\dot{7}$.

Find the series in which

14. The 10th term is 320 and the 6th term 20.

15. The 5th term is $\frac{27}{16}$ and the 9th term is $\frac{1}{3}$.

16. The 7th term is 625 and the 4th term -5.

17. The 3d term is $\frac{9}{16}$ and the 6th term $-4\frac{1}{2}$.

18. Divide 183 into three parts in G. P. such that the sum of the first and third is $2\frac{1}{20}$ times the second.

19. Show that the product of any odd number of consecutive terms of a G. P. will be equal to the nth power of the middle term, n being the number of terms.

20. The first two terms of an infinite G. P. are together equal to 1, and every term is twice the sum of all the terms which follow. Find the series.

Sum the following series:

21. $y^2 + 2b,\ y^4 + 4b,\ y^6 + 6b, \cdots$ to n terms.

22. $\dfrac{3+2\sqrt{2}}{3-2\sqrt{2}},\ 1,\ \dfrac{3-2\sqrt{2}}{3+2\sqrt{2}}, \cdots$ to infinity.

23. $\sqrt{\frac{3}{2}},\ \frac{1}{3}\sqrt{2},\ \frac{2}{9}\sqrt{\frac{2}{3}}$, to infinity.

24. $2n - \frac{1}{2},\ 4n + \frac{1}{6},\ 6n - \frac{1}{18}, \cdots$ to $2n$ terms.

25. The sum of four numbers in G. P. is equal to the common ratio plus 1, and the first term is $\frac{1}{17}$. Find the numbers.

26. The difference between the first and second of four numbers in G. P. is 96, and the difference between the third and fourth is 6. Find the numbers.

27. The sum of \$225 was divided among four persons in such a manner that the shares were in G. P., and the difference between the greatest and least was to the difference between the means as 21 to 6. Find the share of each.

28. The sum of three numbers in G. P. is 13, and the sum of their reciprocals is $\frac{13}{9}$. Find the numbers.

HARMONICAL PROGRESSION.

383. Definition. Three quantities, a, b, c, are said to be in **Harmonical Progression** when $\dfrac{a}{c} = \dfrac{a-b}{b-c}$.

Any number of quantities are said to be in Harmonical Progression when every three consecutive terms are in Harmonical Progression.

384. The Reciprocals of Quantities in Harmonical Progression are in Arithmetical Progression.

By definition, if a, b, c are in Harmonical Progression,

$$\frac{a}{c} = \frac{a-b}{b-c};$$

$$\therefore\ a(b-c) = c(a-b),$$

dividing every term by abc,

$$\frac{1}{c} - \frac{1}{b} = \frac{1}{b} - \frac{1}{a},$$

which proves the proposition. We may therefore define an Harmonical Progression as *a series of quantities the reciprocals of which are in Arithmetical Progression.*

385. Solution of Questions in H. P. Harmonical properties are chiefly interesting because of their importance in Geometry and in the Theory of Sound: in Algebra the proposition just proved is the only one of any importance. There is no general formula for the sum of any number of quantities in Harmonical Progression. Questions in H. P. are generally solved by inverting the terms, and making use of the properties of the corresponding A. P.

Ex. The 12th term of an H. P is $\frac{1}{5}$, and the 19th term is $\frac{3}{22}$: find the series.

Let a be the first term, d the common difference of the corresponding A. P.; then

$$5 = \text{the 12th term} = a + 11d;$$

and $$\tfrac{22}{3} = \text{the 19th term} = a + 18d;$$

whence $$d = \tfrac{1}{3},\ a = \tfrac{4}{3}.$$

Hence the Arithmetical Progression is $\frac{4}{3}, \frac{5}{3}, 2, \frac{7}{3}, \cdots$

and the Harmonical Progression is $\frac{3}{4}, \frac{3}{5}, \frac{1}{2}, \frac{3}{7}, \cdots$.

386. Harmonic Mean. When three quantities are in Harmonic Progression the middle one is said to be the *Harmonic Mean* of the other two.

387. To find the harmonic mean between two given quantities.

Let a, b be the two quantities, H their harmonic mean; then $\frac{1}{a}$, $\frac{1}{H}$, $\frac{1}{b}$ are in A. P.;

$$\therefore \frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H},$$

$$\frac{2}{H} = \frac{1}{a} + \frac{1}{b},$$

$$H = \frac{2ab}{a+b}.$$

388. Relation between the Arithmetic, Geometric, and Harmonic Means. If A, G, H be the arithmetic, geometric, and harmonic means between a and b, we have proved

$$A = \frac{a+b}{2} \quad \ldots \ldots \ldots \quad (1).$$

$$G = \sqrt{ab}. \quad \ldots \ldots \ldots \quad (2).$$

$$H = \frac{2ab}{a+b} \quad \ldots \ldots \ldots \quad (3).$$

Therefore

$$AH = \frac{a+b}{2} \cdot \frac{2ab}{a+b}$$

$$= ab = G^2;$$

that is, G is the geometric mean between A and H.

389. Miscellaneous Questions in the Progressions. Miscellaneous questions in the Progressions afford scope for much skill and ingenuity, the solution being often very neatly effected by some special artifice. The student will find the following hints useful.

1. If the same quantity be added to, or subtracted from, all the terms of an A. P., the resulting terms will form an A. P., with the same common difference as before. [Art. 365.]

2. If all the terms of an A. P. be multiplied or divided by the same quantity, the resulting terms form an A. P., but with a new common difference. [Art. 365.]

3. If all the terms of a G. P. be multiplied or divided by the same quantity, the resulting terms form a G. P. with the same common ratio as before. [Art. 374.]

4. If $a, b, c, d \cdots$ be in G. P., they are also in *continued proportion*, since by definition

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\cdots=\frac{1}{r},$$

Conversely, a series of quantities in continued proportion may be represented by $x, xr, xr^2, \cdots$.

Ex. 1. Find three quantities in G. P. such that their product is 343, and their sum $30\frac{1}{3}$.

Let $\frac{a}{r}$, a, ar be the three quantities;

then we have $$\frac{a}{r}\times a \times ar = 343 \quad \ldots\ldots\ldots \quad (1),$$

and $$a\left(\frac{1}{r}+1+r\right)=\frac{91}{3} \quad \ldots\ldots\ldots \quad (2).$$

From (1) $$a^3 = 343;$$

$$\therefore\ a = 7;$$

hence from (2) $$7(1+r+r^2)=\tfrac{91}{3}r.$$

Whence we obtain $$r = 3, \text{ or } \tfrac{1}{3};$$

and the numbers are $\frac{7}{3}$, 7, 21.

Ex. 2. If a, b, c be in H. P., prove that $\frac{a}{b+c}$, $\frac{b}{c+a}$, $\frac{c}{a+b}$ are also in H. P.

Since $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$ are in A. P.,

$$\frac{a+b+c}{a},\ \frac{a+b+c}{b},\ \frac{a+b+c}{c} \text{ are in A. P.};$$

$$\therefore\ 1+\frac{b+c}{a},\ 1+\frac{a+c}{b},\ 1+\frac{a+b}{c} \text{ are in A. P.};$$

$$\therefore\ \frac{b+c}{a},\ \frac{a+c}{b},\ \frac{a+b}{c} \text{ are in A. P.};$$

$$\therefore\ \frac{a}{b+c},\ \frac{b}{c+a},\ \frac{c}{a+b} \text{ are in H. P.}$$

Ex. 3. The nth term of an A. P. is $\frac{n}{5}+2$: find the sum of 49 terms.

Let a be the first term, and l the last; then by putting $n=1$, and $n=49$ respectively, we obtain

$$a=\tfrac{1}{5}+2, \quad l=\tfrac{49}{5}+2;$$

$$\therefore\ S=\frac{n}{2}(a+l)=\tfrac{49}{2}\left(\tfrac{50}{5}+4\right)$$

$$=\tfrac{49}{2}\times 14=343.$$

Ex. 4. If a, b, c, d, e be in G. P., prove that $b+d$ is the geometric mean between $a+c$ and $c+e$.

Since a, b, c, d, e are in continued proportion,

$$\frac{a}{b}=\frac{b}{c}=\frac{c}{d}=\frac{d}{e};$$

$\therefore$ each ratio $$=\frac{a+c}{b+d}=\frac{b+d}{c+e}. \qquad \text{[Art. 347.]}$$

Whence $$(b+d)^2=(a+c)(c+e).$$

EXAMPLES XXXIV. e.

1. Find the 6th term of the series $4,\ 2,\ 1\frac{1}{3}, \cdots$.
2. Find the 21st term of the series $2\frac{1}{2},\ 1\frac{12}{13},\ 1\frac{9}{16}, \cdots$.
3. Find the 8th term of the series $1\frac{1}{3},\ 1\frac{11}{17},\ 2\frac{2}{13}, \cdots$.
4. Find the nth term of the series $3,\ 1\frac{1}{2},\ 1, \cdots$.

Find the series in which

5. The 15th term is $\frac{1}{25}$, and the 23d term is $\frac{1}{41}$.
6. The 2d term is 2, and the 31st term is $\frac{4}{31}$.
7. The 39th term is $\frac{1}{11}$, and the 54th term is $\frac{1}{26}$.

Find the harmonic mean between

8. 2 and 4.
9. 1 and 13.
10. $\frac{1}{4}$ and $\frac{1}{10}$.
11. $\frac{1}{a}$ and $\frac{1}{b}$.
12. $\frac{1}{x+y}$ and $\frac{1}{x-y}$.
13. $x+y$ and $x-y$.
14. Insert two harmonic means between 4 and 12.
15. Insert three harmonic means between $2\frac{2}{5}$ and 12.
16. Insert four harmonic means between 1 and 6.

17. If G be the geometric mean between two quantities A and B, show that the ratio of the arithmetic and harmonic means of A and G is equal to the ratio of the arithmetic and harmonic means of G and B.

18. To each of three consecutive terms of a G. P., the second of the three is added. Show that the three resulting quantities are in H. P.

Sum the following series:

19. $1 + 1\frac{3}{4} + 3\frac{1}{16} + \cdots$ to 6 terms.

20. $1 + 1\frac{3}{4} + 2\frac{1}{2} + \cdots$ to 6 terms.

21. $(2a + x) + 3a + (4a - x) + \cdots$ to p terms.

22. $1\frac{4}{5} - 1\frac{1}{5} + \frac{4}{5} - \cdots$ to 8 terms.

23. $1\frac{4}{5} + 1\frac{1}{5} + \frac{3}{5} + \cdots$ to 12 terms.

24. If $x - a$, $y - a$, and $z - a$ be in G.P., prove that $2(y - a)$ is the harmonic mean between $y - x$ and $y - z$.

25. If a, b, c, d be in A. P., a, e, f, d in G. P., a, g, h, d in H. P. respectively; prove that $ad = ef = bh = cg$.

26. If a^2, b^2, c^2 be in A. P., prove that $b+c$, $c+a$, $a+b$ are in H. P.

CHAPTER XXXV.

Permutations and Combinations.

390. Each of the *arrangements* which can be made by taking some or all of a number of things is called a **permutation.**

Each of the *groups* or *selections* which can be made by taking some or all of a number of things is called a **combination.**

Thus the *permutations* which can be made by taking the letters a, b, c, d two at a time are twelve in number; namely,

$$ab, \quad ac, \quad ad, \quad bc, \quad bd, \quad cd,$$
$$ba, \quad ca, \quad da, \quad cb, \quad db, \quad dc;$$

each of these presenting a different *arrangement* of two letters.

The *combinations* which can be made by taking the letters a, b, c, d two at a time are six in number; namely,

$$ab, \quad ac, \quad ad, \quad bc, \quad bd, \quad cd;$$

each of these presenting a different *selection* of two letters.

From this it appears that in forming *combinations* we are only concerned with the *number* of things each selection contains; whereas in forming *permutations* we have also to consider the *order* of the things which make up each arrangement; for instance, if from four letters a, b, c, d we make a selection of three, such as abc, this single combination admits of being arranged in the following ways:

$$abc, \quad acb, \quad bca, \quad bac, \quad cab, \quad cba,$$

and so gives rise to six different permutations.

391. Fundamental Principle. Before discussing the general propositions of this chapter the following important principle should be carefully noticed.

If one operation can be performed in m ways, and (when it has been performed in any one of these ways) a second operation can then be performed in n ways; the number of ways of performing the two operations will be $m \times n$.

If the first operation be performed in *any one* way, we can associate with this any of the n ways of performing the second operation; and thus we shall have n ways of performing the two operations without considering more than *one* way of performing the first; and so, corresponding to *each* of the m ways of performing the first operation, we shall have n ways of performing the two; hence the product $m \times n$ represents the total number of ways in which the two operations can be performed.

Ex. Suppose there are 10 steamers plying between New York and Liverpool: in how many ways can a man go from New York to Liverpool and return by a different steamer?

There are *ten* ways of making the first passage; and with each of these there is a choice of *nine* ways of returning (since the man is not to come back by the same steamer); hence the number of ways of making the two journeys is 10×9, or 90.

This principle may easily be extended to the case in which there are more than two operations each of which can be performed in a given number of ways.

Ex. Three travellers arrive at a town where there are four hotels; in how many ways can they take up their quarters, each at a different hotel?

The first traveller has choice of four hotels, and when he has made his selection in any one way, the second traveller has a choice of three; therefore the first two can make their choice in 4×3 ways; and with any one such choice the third traveller can select his hotel in 2 ways; hence the required number of ways is $4 \times 3 \times 2$, or 24.

392. To find the number of permutations of *n* dissimilar things taken *r* at a time.

This is the same thing as finding the number of ways in which we can fill r places when we have n different things at our disposal.

The first place may be filled in n ways, for any one of the n things may be taken; when it has been filled in any one of these ways, the second place can then be filled in $n-1$ ways; and since each way of filling the first place can be associated with each way of filling the second, the number of ways in which the first two places can be filled is given by the product $n(n-1)$. And when the first two places have been filled in any way, the third place can be filled in $n-2$ ways. And reasoning as before, the number of ways in which three places can be filled is $n(n-1)(n-2)$.

Proceeding thus, and noticing that *a new factor is introduced with each new place filled*, and that at any stage *the number of factors is the same as the number of places filled*, we shall have the number of ways in which r places can be filled equal to

$$n(n-1)(n-2)\cdots \text{ to } r \text{ factors.}$$

We here see that each factor is formed by taking from n a number *one less* than that which applies to the place filled by that factor; hence the rth factor is $n-(r-1)$, or $n-r+1$.

Therefore the number of permutations of n things taken r at a time is

$$n(n-1)(n-2)\cdots(n-r+1).$$

Cor. The number of permutations of n things taken all at a time is

$$n(n-1)(n-2)\cdots \text{ to } n \text{ factors,}$$

or $$n(n-1)(n-2)\cdots 3\cdot 2\cdot 1.$$

It is usual to denote this product by the symbol $\underline{|n}$, which is read "factorial n." Also $n!$ is sometimes used for $\underline{|n}$.

393. We shall in future denote the number of permutations of n things taken r at a time by the symbol ${}^nP_r$, so that

$${}^nP_r = n(n-1)(n-2)\cdots(n-r+1);$$

also $${}^nP_n = \underline{|n}.$$

In working numerical examples it is useful to notice that the suffix in the symbol ${}^nP_r$ always denotes the number of factors in the formula we are using.

Ex. 1. Four persons enter a carriage in which there are six seats: in how many ways can they take their places?

The first person may seat himself in 6 ways; and then the second person in 5; the third in 4; and the fourth in 3; and since each of these ways may be associated with each of the others, the required answer is $6 \times 5 \times 4 \times 3$, or 360.

Ex. 2. How many different numbers can be formed by using six out of the nine digits 1, 2, 3, $\cdots$ 9?

Here we have 9 different things, and we have to find the number of permutations of them taken 6 at a time;

$$\therefore \text{ the required result} = {}^9P_6$$
$$= 9 \times 8 \times 7 \times 6 \times 5 \times 4 = 60480.$$

394. To find the number of combinations of *n* dissimilar things taken *r* at a time.

Let ${}^nC_r$ denote the required number of combinations.

Then each of these combinations consists of a group of r dissimilar things which can be arranged among themselves in $\lfloor r$ ways. [Art. 392, Cor.]

Hence ${}^nC_r \times \lfloor r$ is equal to the number of *arrangements* of n things taken r at a time; that is,

$${}^nC_r \times \lfloor r = {}^nP_r = n(n-1)(n-2)\cdots(n-r+1);$$
$$\therefore \ {}^nC_r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r} \quad . \quad . \quad (1).$$

Cor. This formula for ${}^nC_r$ may also be written in a different form; for if we multiply the numerator and the denominator by $\lfloor n-r$ we obtain

$$\frac{n(n-1)(n-2)\cdots(n-r+1) \times \lfloor n-r}{\lfloor r \lfloor n-r}, \text{ or } \frac{\lfloor n}{\lfloor r \lfloor n-r} \quad . \quad (2);$$

since $n(n-1)(n-2)\cdots(n-r+1) \times \lfloor n-r = \lfloor n$.

It will be convenient to remember both these expressions for ${}^nC_r$, using (1) in all cases where a numerical result is required, and (2) when it is sufficient to leave it in an algebraic shape.

NOTE. If in formula (2) we put $r = n$, we have

$$^nC_n = \frac{\underline{|n}}{\underline{|n}\,\underline{|0}} = \frac{1}{\underline{|0}};$$

but $^nC_n = 1$, so that if the formula is to be true for $r = n$, the symbol $\underline{|0}$ must be considered as equivalent to 1.

Ex. From 12 books in how many ways can a selection of 5 be made, (1) when one specified book is always included, (2) when one specified book is always excluded?

(1) Since the specified book is to be included in every selection, we have only to choose 4 out of the remaining 11.

Hence the number of ways $= {}^{11}C_4 = \dfrac{11 \times 10 \times 9 \times 8}{1 \times 2 \times 3 \times 4} = 330.$

(2) Since the specified book is always to be excluded, we have to select the 5 books out of the remaining 11.

Hence the number of ways $= {}^{11}C_5 = \dfrac{11 \times 10 \times 9 \times 8 \times 7}{1 \times 2 \times 3 \times 4 \times 5} = 462.$

395. The number of combinations of *n* things *r* at a time is equal to the number of combinations of *n* things *n − r* at a time.

In making all the possible combinations of n things, to each group of r things we select, there is left a corresponding group of $n - r$ things; that is, the number of combinations of n things r at a time is the same as the number of combinations of n things $n - r$ at a time;

$$\therefore\ {}^nC_r = {}^nC_{n-r}.$$

This result is frequently useful in enabling us to abridge arithmetical work.

Ex. Out of 14 men in how many ways can an eleven be chosen?

The required number $= {}^{14}C_{11} = {}^{14}C_3 = \dfrac{14 \times 13 \times 12}{1 \times 2 \times 3} = 364.$

If we had made use of the formula ${}^{14}C_{11}$, we should have had to reduce an expression whose numerator and denominator each contained 11 factors.

396. In the examples which follow it is important to notice that the formula for *permutations* should not be used until the suitable *selections* required by the question have been made.

Ex. 1. From 7 Englishmen and 4 Americans a committee of 6 is to be formed: in how many ways can this be done, (1) when the committee contains exactly 2 Americans, (2) at least 2 Americans?

(1) The number of ways in which the Americans can be chosen is ${}^4C_2$; and the number of ways in which the Englishmen can be chosen is ${}^7C_4$. Each of the first groups can be associated with each of the second; hence

$$\text{the required number of ways} = {}^4C_2 \times {}^7C_4$$
$$= \frac{\lfloor\underline{4}}{\lfloor\underline{2}\,\lfloor\underline{2}} \times \frac{\lfloor\underline{7}}{\lfloor\underline{4}\,\lfloor\underline{3}} = \frac{\lfloor\underline{7}}{\lfloor\underline{2}\,\lfloor\underline{2}\,\lfloor\underline{3}} = 210.$$

(2) We exhaust all the suitable combinations by forming all the groups containing 2 Americans and 4 Englishmen; then 3 Americans and 3 Englishmen; and lastly 4 Americans and 2 Englishmen.

The *sum* of the three results gives the answer. Hence the required number of ways $= {}^4C_2 \times {}^7C_4 + {}^4C_3 \times {}^7C_3 + {}^4C_4 \times {}^7C_2$

$$= \frac{\lfloor\underline{4}}{\lfloor\underline{2}\,\lfloor\underline{2}} \times \frac{\lfloor\underline{7}}{\lfloor\underline{4}\,\lfloor\underline{3}} + \frac{\lfloor\underline{4}}{\lfloor\underline{3}} \times \frac{\lfloor\underline{7}}{\lfloor\underline{3}\,\lfloor\underline{4}} + 1 \times \frac{\lfloor\underline{7}}{\lfloor\underline{2}\,\lfloor\underline{5}}$$
$$= 210 + 140 + 21 = 371.$$

In this example we have only to make use of the suitable formula for *combinations*, for we are not concerned with the possible arrangements of the members of the committee among themselves.

Ex. 2. Out of 7 consonants and 4 vowels, how many words can be made each containing 3 consonants and 2 vowels?

The number of ways of choosing the three consonants is ${}^7C_3$, and the number of ways of choosing the two vowels is ${}^4C_2$; and since each of the first groups can be associated with each of the second, the number of combined groups, each containing 3 consonants and 2 vowels, is ${}^7C_3 \times {}^4C_2$.

Further, each of these groups contains 5 letters, which may be arranged among themselves in $\lfloor\underline{5}$ ways. Hence

$$\text{the required number of words} = \frac{\lfloor\underline{7}}{\lfloor\underline{3}\,\lfloor\underline{4}} \times \frac{\lfloor\underline{4}}{\lfloor\underline{2}\,\lfloor\underline{2}} \times \lfloor\underline{5}$$
$$= 5 \times \lfloor\underline{7} = 25200.$$

EXAMPLES XXXV. a.

1. Find the value of ${}^5P_4$, ${}^7P_6$, ${}^8C_5$, ${}^{25}C_{23}$.

2. How many different arrangements can be made by taking (1) five, (2) all of the letters of the word *soldier*?

3. If ${}^nC_3 : {}^{n-1}C_4 = 8 : 5$, find n.

4. How many different selections of four coins can be made from a bag containing a dollar, a half-dollar, a quarter, a florin, a shilling, a franc, a dime, a sixpence, and a penny ?

5. How many numbers between 3000 and 4000 can be made with the digits 9, 3, 4, 6 ?

6. In how many ways can the letters of the word *volume* be arranged if the vowels can only occupy the even places ?

7. If the number of permutations of n things four at a time is fourteen times the number of permutations of $n-2$ things three at a time, find n.

8. From 5 teachers and 10 boys how many committees can be selected containing 3 teachers and 6 boys ?

9. If ${}^{20}C_r = {}^{20}C_{r-10}$, find ${}^{r}C_{12}$, ${}^{18}C_r$.

10. Out of the twenty-six letters of the alphabet in how many ways can a word be made consisting of five different letters two of which must be a and e ?

11. How many words can be formed by taking 3 consonants and 2 vowels from an alphabet containing 21 consonants and 5 vowels ?

12. A stage will accommodate 5 passengers on each side : in how many ways can 10 persons take their seats when two of them remain always upon one side and a third upon the other ?

397. Hitherto, in the formulae we have proved, the things have been regarded as *unlike.* Before considering cases in which some one or more sets of things may be *like*, it is necessary to point out exactly in what sense the words *like* and *unlike* are used. When we speak of things being *dissimilar, different, unlike,* we imply that the things are *visibly unlike,* so as to be easily distinguishable from each other. On the other hand, we shall always use the term *like* things to denote such as are alike to the eye and cannot be distinguished from each other. For instance, in Ex. 2, Art. 396, the consonants and the vowels may be said each to consist of a group of things united by a common characteristic, and thus in a certain sense to be of the same kind; but they cannot be regarded as like things, because there is an individuality existing among the things of each group which makes them easily distinguishable from each other. Hence, in the final stage of the example we considered each group to consist of five *dissimilar* things and therefore capable of $\underline{|5}$ arrangements among themselves. [Art. 392, Cor.]

398. To find the permutations of *n* things, taking them all at a time, when *p* things are of one kind, *q* of another kind, *r* of a third kind, and the rest all different.

Let there be n letters; suppose p of them to be a, q of them to be b, r of them to be c, and the rest to be unlike.

Let x be the required number of permutations; then if in *any one* of these permutations the p letters a were replaced by p unlike letters different from any of the rest, from this single permutation, without altering the position of any of the remaining letters, we could form $\underline{|p}$ new permutations. Hence if this change were made in each of the x permutations, we should obtain $x \times \underline{|p}$ permutations.

Similarly, if the q letters b were replaced by q unlike letters, the number of permutations would be $x \times \underline{|p} \times \underline{|q}$.

In like manner, by replacing the r letters c by r unlike letters, we should finally obtain $x \times \underline{|p} \times \underline{|q} \times \underline{|r}$ permutations.

But the things are now all different, and therefore admit of $\underline{|n}$ permutations among themselves. Hence

$$x \times \underline{|p} \times \underline{|q} \times \underline{|r} = \underline{|n};$$

that is,

$$x = \frac{\underline{|n}}{\underline{|p}\,\underline{|q}\,\underline{|r}};$$

which is the required number of permutations.

Any case in which the things are not all different may be treated similarly.

Ex. 1. How many different permutations can be made out of the letters of the word *assassination* taken all together?

We have here 13 letters of which 4 are s, 3 are a, 2 are i, and 2 are n. Hence the number of permutations

$$= \frac{\underline{|13}}{\underline{|4}\,\underline{|3}\,\underline{|2}\,\underline{|2}}$$

$$= 13 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 3 \cdot 5$$

$$= 1001 \times 10800 = 10810800.$$

Ex. 2. How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1, so that the odd digits always occupy the odd places?

The odd digits 1, 3, 3, 1 can be arranged in their four places in

$$\frac{\lfloor 4}{\lfloor 2 \, \lfloor 2} \text{ ways} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The even digits 2, 4, 2 can be arranged in their three places in

$$\frac{\lfloor 3}{\lfloor 2} \text{ ways} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Each of the ways in (1) can be associated with each of the ways in (2).

Hence the required number $= \dfrac{\lfloor 4}{\lfloor 2 \, \lfloor 2} \times \dfrac{\lfloor 3}{\lfloor 2} = 6 \times 3 = 18.$

399. To find the number of permutations of *n* things *r* at a time, when each thing may be repeated once, twice, ... up to *r* times in any arrangement.

Here we have to consider the number of ways in which r places can be filled when we have n different things at our disposal, each of the n things being used as often as we please in any arrangement.

The first place may be filled in n ways, and, when it has been filled in any one way, the second place may also be filled in n ways, since we are not precluded from using the same thing again. Therefore the number of ways in which the first two places can be filled is $n \times n$ or n^2.

The third place can also be filled in n ways, and therefore the first three places in n^3 ways.

Proceeding thus, and noticing that at any stage the index of n is always the same as the number of places filled, we shall have the number of ways in which the r places can be filled equal to n^r.

Ex. In how many ways can 5 prizes be given away to 4 boys, when each boy is eligible for all the prizes?

Any one of the prizes can be given in 4 ways; and then any one of the remaining prizes can also be given in 4 ways, since it may be obtained by the boy who has already received a prize. Thus two prizes can be given away in 4^2 ways, three prizes in 4^3 ways, and so on. Hence the 5 prizes can be given away in 4^5, or 1024 ways.

400. **To find the total number of ways in which it is possible to make a selection by taking some or all of *n* things.**

Each thing may be dealt with in two ways, for it may either be taken or left; and since either way of dealing with any one thing may be associated with either way of dealing with each one of the others, the number of ways of dealing with the n things is

$$2 \times 2 \times 2 \times 2 \ldots \text{ to } n \text{ factors.}$$

But this includes the case in which all the things are left, therefore, rejecting this case, the total number of ways is $2^n - 1$.

This is often spoken of as "the total number of combinations" of n things.

Ex. A man has 6 friends; in how many ways may he invite one or more of them to dinner?

He has to select some or all of his 6 friends; and therefore the number of ways is $2^6 - 1$, or 63.

This result can be verified in the following manner.

The guests may be invited singly, in twos, threes, ...; therefore the number of selections

$$= {}^6C_1 + {}^6C_2 + {}^6C_3 + {}^6C_4 + {}^6C_5 + {}^6C_6$$
$$= 6 + 15 + 20 + 15 + 6 + 1 = 63.$$

401. **To find for what value of *r* the number of combinations of *n* things *r* at a time is greatest.**

Since $${}^nC_r = \frac{n(n-1)(n-2)\cdots(n-r+2)(n-r+1)}{1\cdot 2\cdot 3\cdots(r-1)r},$$

and $${}^nC_{r-1} = \frac{n(n-1)(n-2)\cdots(n-r+2)}{1\cdot 2\cdot 3\cdots(r-1)};$$

$$\therefore\ {}^nC_r = {}^nC_{r-1} \times \frac{n-r+1}{r}.$$

The multiplying factor $\frac{n-r+1}{r}$ may be written $\frac{n+1}{r} - 1$, which shows that it decreases as r increases. Hence as r receives the values $1, 2, 3, \cdots$ in succession, ${}^nC_r$ is continually increased, until $\frac{n+1}{r}-1$ becomes equal to 1 or less than 1.

Now $\frac{n+1}{r}-1>1$, so long as $\frac{n+1}{r}>2$; that is, $\frac{n+1}{2}>r$.

We have to choose the greatest value of r consistent with this inequality.

(1) Let n be even, and equal to $2m$; then

$$\frac{n+1}{2}=\frac{2m+1}{2}=m+\tfrac{1}{2};$$

and for all values of r up to m inclusive this is greater than r. Hence by putting $r=m=\frac{n}{2}$, we find that the greatest number of combinations is ${}^nC_{\frac{n}{2}}$.

(2) Let n be odd, and equal to $2m+1$; then

$$\frac{n+1}{2}=\frac{2m+2}{2}=m+1;$$

and for all values of r up to m inclusive this is greater than r; but when $r=m+1$, the multiplying factor becomes equal to 1, and

$${}^nC_{m+1}={}^nC_m;\ \text{that is, } {}^nC_{\frac{n+1}{2}}={}^nC_{\frac{n-1}{2}};$$

and therefore the number of combinations is greatest when the things are taken $\frac{n+1}{2}$, or $\frac{n-1}{2}$ at a time; the result being the same in the two cases.

EXAMPLES XXXV. b.

1. Find the number of permutations which can be made from all the letters of the words,

(1) *irresistible*, (2) *phenomenon*, (3) *tittle-tattle*.

2. How many different numbers can be formed by using the seven digits 2, 3, 4, 3, 3, 1, 2? How many with the digits 2, 3, 4, 3, 3, 0, 2?

3. How many words can be formed from the letters of the word *Simoom*, so that vowels and consonants occur alternately in each word?

4. A telegraph has 5 arms, and each arm has 4 distinct positions, including the position of rest: find the total number of signals that can be made.

5. In how many ways can n things be given to m persons, when there is no restriction as to the number of things each may receive?

6. How many different arrangements can be made out of the letters of the expression $a^5b^3c^6$ when written at full length?

7. There are 4 copies each of 3 different volumes; find the number of ways in which they can be arranged on one shelf.

8. In how many ways can 6 persons form a ring? Find the number of ways in which 4 gentlemen and 4 ladies can sit at a round table so that no two gentlemen sit together.

9. In how many ways can a word of 4 letters be made out of the letters a, b, e, c, d, o, when there is no restriction as to the number of times a letter is repeated in each word?

10. How many arrangements can be made out of the letters of the word *Toulouse*, so that the consonants occupy the first, fourth, and seventh places?

11. A boat's crew consists of eight men of whom one can only row on bow side and one only on stroke side: in how many ways can the crew be arranged?

12. Show that ${}^{n+1}C_r = {}^{n}C_r + {}^{n}C_{r-1}$.

13. If ${}^{2n}C_3 : {}^{n}C_2 = 44 : 3$, find n.

14. Out of the letters A, B, C, p, q, r, how many arrangements can be made beginning with a capital?

15. Find the number of combinations of 50 things 46 at a time.

16. If ${}^{18}C_r = {}^{18}C_{r+2}$, find ${}^{r}C_5$.

17. In how many ways is it possible to draw a sum of money from a bag containing a dollar, a half-dollar, a quarter, a dime, a five-cent piece, a two-cent piece, and a penny?

CHAPTER XXXVI.

PROBABILITY (CHANCE).

402. DEFINITION. If an event can happen in a ways and fail in b ways, and each of these ways is equally likely, the **probability**, or the **chance**, of its *happening* is $\frac{a}{a+b}$, and of its *failing* is $\frac{b}{a+b}$. Hence to find the probability of an event happening, *divide the number of favorable ways by the whole number of ways favorable and unfavorable.*

For instance, if in a lottery there are 7 prizes and 25 blanks, the chance that a person holding 1 ticket will win a prize is $\frac{7}{32}$, and his chance of not winning is $\frac{25}{32}$.

Instead of saying that the chance of the happening of an event is $\frac{a}{a+b}$, it is sometimes stated that *the odds are a to b in favor of the event, or b to a against the event.*

Thus in the above the odds are seven to twenty-five in favor of the drawing of a prize, and twenty-five to seven against success.

403. The reason for the mathematical definition of probability may be made clear by the following considerations:

If an event can happen in a ways and fail to happen in b ways, and all these ways are equally likely, we can assert that the chance of its happening is to the chance of its failing as a to b. Thus if the chance of its happening is represented by ka, where k is an undetermined constant, then the chance of its failing will be represented by kb.

$$\therefore \text{ chance of happening } + \text{ chance of failing } = k(a+b).$$

Now the event is certain to happen or to fail; therefore the sum of the chances of happening and failing must represent *certainty*. If therefore we agree to take *certainty as our unit*, we have

$$1 = k(a+b), \text{ or } k = \frac{1}{a+b},$$

$\therefore$ the chance that the event will happen is $\dfrac{a}{a+b}$,

and the chance that the event will not happen is $\dfrac{b}{a+b}$.

Cor. If p is the probability of the happening of an event, the probability of its not happening is $1-p$.

404. The definition of probability in Art. 402 may be given in a slightly different form which is sometimes useful. If c is the total number of cases, each being equally likely to occur, and of these a are favorable to the event, then the probability that the event will happen is $\dfrac{a}{c}$, and the probability that it will not happen is $1 - \dfrac{a}{c}$.

Ex. 1. (*a*) From a bag containing 4 white and 5 black balls a man draws a single ball at random. What is the chance that it is black?

A black ball can be drawn in 5 ways, since any one of the 5 black balls may be drawn. In the same way any one of the 4 white balls may be drawn.

Hence the chance of drawing a black ball is $\dfrac{5}{4+5}$, or $\dfrac{5}{9}$.

(*b*) Suppose the man draws 3 balls at random. What are the odds against these being all black?

The total number of ways in which 3 balls can be drawn is ${}^9C_3$, and the total number of ways of drawing 3 black balls is ${}^5C_3$; therefore the chance of drawing 3 black balls

$$= \frac{{}^5C_3}{{}^9C_3} = \frac{5 \cdot 4 \cdot 3}{9 \cdot 8 \cdot 7} = \frac{5}{42}.$$

Thus the odds against the event are 37 to 5.

Ex. 2. From a bag containing 5 red balls, 4 white balls, and 5 black balls, 6 balls are drawn at random. What is the chance that 3 are white, 2 black, and 1 red?

The number of combinations of 4 white balls, taken 3 at a time, is $\frac{4 \cdot 3 \cdot 2}{1 \cdot 2 \cdot 3}$ or 4. In the same manner the number of combinations of 5 black balls, taken 2 at a time, is $\frac{5 \cdot 4}{1 \cdot 2}$ or 10. Since each of the 4 combinations of white balls may be taken with any one of the 10 combinations of black, and with each of the combinations so formed we may take any one of the 5 red balls, the total number of combinations will be $4 \cdot 10 \cdot 5$ or 200. But the number of combinations of the entire number of balls, taken 6 at a time is $\frac{14 \cdot 13 \cdot 12 \cdot 11 \cdot 10 \cdot 9}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}$ or 3003, hence the chance that 3 white, 2 black, and 1 red ball will be drawn at one time is $\frac{200}{3003}$.

Ex. 3. A has 3 shares in a lottery in which there are 3 prizes and 6 blanks; B has 1 share in a lottery in which there is 1 prize and 2 blanks. Show that A's chance of success is to B's as 16 to 7.

A may draw 3 prizes in 1 way; he may draw 2 prizes and 1 blank in $\frac{3 \cdot 2}{1 \cdot 2} \times 6$ ways; he may draw 1 prize and 2 blanks in $3 \times \frac{6 \cdot 5}{1 \cdot 2}$ ways; the sum of these numbers is 64, which is the number of ways in which A can win a prize. Also he can draw 3 tickets in $\frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3}$, or 84 ways; therefore A's chance of success $= \frac{64}{84} = \frac{16}{21}$.

B's chance of success is clearly $\frac{1}{3}$; therefore A's chance : B's chance $= \frac{16}{21} : \frac{1}{3} = 16 : 7$.

Or we might have reasoned thus: A will get *all blanks* in $\frac{6 \cdot 5 \cdot 4}{1 \cdot 2 \cdot 3}$, or 20 ways; the chance of which is $\frac{20}{84}$, or $\frac{5}{21}$; therefore A's chance of success $= 1 - \frac{5}{21} = \frac{16}{21}$.

405. From the examples given it will be seen that the solution of the easier kinds of questions in Probability requires nothing more than a knowledge of the definition of Probability, and the application of the laws of Permutations and Combinations.

EXAMPLES XXXVI.

1. A bag contains 5 white, 7 black, and 4 red balls; find the chance of drawing: (*a*) One white ball; (*b*) Two white balls; (*c*) Three white balls; (*d*) One ball of each color; (*e*) One white, two black, and three red balls.

2. If four coins are tossed, find the chance that there should be 2 heads and 2 tails.

3. One of two events must happen: given that the chance of the one is two-thirds that of the other, find the odds in favor of the other.

4. Thirteen persons take their places at a round table. Show that it is 5 to 1 against 2 particular persons sitting together.

5. There are three events A, B, C, one of which must, and only one can, happen; the odds are 8 to 3 against A, 5 to 2 against B. Find the odds against C.

6. A has 3 shares in a lottery containing 3 prizes and 9 blanks; B has 2 shares in a lottery containing 2 prizes and 6 blanks. Compare their chances of success.

7. There are three works, one consisting of 3 volumes, one of 4, and the other of 1 volume. They are placed on a shelf at random. Prove that the chance that volumes of the same works are all together is $\frac{3}{140}$.

8. The letters forming the word *Clifton* are placed at random in a row. What is the chance that the two vowels come together?

9. In a hand at whist what is the chance that the four kings are held by a specified player.

10. There are 4 dollars and 3 half-dollars placed at random in a line. Show that the chance of the extreme coins being both half-dollars is $\frac{1}{7}$.

MISCELLANEOUS EXAMPLES VI.

1. Simplify $\dfrac{b-c}{a^2-(b-c)^2}+\dfrac{c-a}{b^2-(c-a)^2}+\dfrac{a-b}{c^2-(a-b)^2}$.

2. Extract the square root of

(i.) $4x^4+6x^3+\frac{89}{4}x^2+15x+25$.

(ii.) $x^8-\dfrac{2x^{11}}{a^3}+2a^4x^4+\dfrac{x^{14}}{a^6}-2ax^7+a^8$.

3. A number of 3 digits exceeds 25 times the sum of the digits by 9; the middle digit increased by 3 is equal to the sum of the other digits, and the unit digit increased by 6 is equal to twice the sum of the other 2 digits: find the number.

4. Find the value of

$$2\sqrt{\tfrac{3}{2}}+3\sqrt{\tfrac{2}{3}}-\left(\tfrac{2}{3}\sqrt[3]{\tfrac{1}{5}}\div\sqrt[3]{\tfrac{25}{4}}\right)\left(\tfrac{1}{2}\sqrt{6}-\sqrt{24}\right).$$

5. Solve (i.) $2=\dfrac{\sqrt{2+x}+\sqrt{2-x}}{\sqrt{2+x}-\sqrt{2-x}}$.

(ii.) $\sqrt{3x-11}+\sqrt{3x}=\sqrt{12x-23}$.

6. Solve $\dfrac{2(x+a)}{x+b}+\dfrac{3(x+b)}{x+a}=5.$

7. The sum of a certain number of terms of an A. P. is 45, and the first and last of these terms are 1 and 17 respectively. Find the number of terms and the common difference of the series.

8. Solve (i.) $\dfrac{2x-3}{x-2}-\dfrac{1}{6}+\dfrac{2x-1}{1-x}=0.$

(ii.) $\sqrt{12x-5}+\sqrt{3x-1}=\sqrt{27x-2}.$

9. Find the value of the seventh term in the expansion of $(a+x)^n$ when $a=\frac{1}{2}$, $x=\frac{1}{3}$, $n=9$.

10. A man starting from A at 11 o'clock passed the fourth milestone at 11.30 and met another man (who started from B at 12) at 12.48; the second man passed the fourth milestone from A at 1.40: find the distance between A and B, and the second man's rate.

11. Show that $x^3+13a^2x>5ax^2+9a^3$, if $x>a$.

12. Extract the cube root of

$$44x^3+63x^2+x^6+27+6x^5+21x^4+54x.$$

13. Solve (i.) $x-y=3,$ $x^2+xy+y^2=93.$ (ii.) $2x^2-9xy+9y^2=5,$ $4x^2-10xy+11y^2=35.$

14. Find a mean proportional between $\dfrac{\sqrt{13-8\sqrt{-3}}\,(a^0+b^0)^{-2}}{ab}$ and the reciprocal of $\dfrac{4-\sqrt{-3}}{16a^3b}$.

15. Two vessels, one of which sails 2 miles an hour faster than the other, start together upon voyages of 1680 and 1152 miles respectively; the slower vessel reaches its destination one day before the other: how many miles per hour did the faster vessel sail?

16. Solve (i.) $x^6=8+7x^3.$

(ii.) $x^{2n}+b^2=c^2-2bx^n.$

17. Two numbers are in the ratio 2 : 7; the numbers obtained by adding 6 to each of the given numbers are in the duplicate ratio of 2 : 3. Find the numbers.

18. Solve (i.) $2bx^2-2b=4x+b^2x.$

(ii.) $\dfrac{x+4}{2x+3}+\dfrac{3x+10}{2x}=\dfrac{2x+3}{x-1}.$

(iii.) $\sqrt{x+3}+\sqrt{x+8}=\sqrt{4x+21}.$

(iv.) $x^2+xy+y=137,$
$y^2+xy+x=205.$

19. Simplify $$\frac{\left[\sqrt{\frac{2+\sqrt{-2}}{2}}-\sqrt{\frac{2-\sqrt{-2}}{2}}\right]^2}{2-\sqrt{6}}.$$

20. Find the sides of a rectangle the area of which is unaltered if its length be increased by 2 feet while its breadth is diminished by 1 foot, and which loses $\frac{4}{7}$ of its area if its length be diminished by 2 feet and its breadth by 4 feet.

21. The first term of a G. P. exceeds the second term by 1, and the sum to infinity is 81: find the series.

22. Find the number of permutations which can be made from all the letters of the word *Mississippi.*

23. Solve (i.) $\sqrt{x+2}+\sqrt{4x+1}-\sqrt{9x+7}=0.$

(ii.) $\dfrac{2x-3}{\sqrt{x-2}+1}=2\sqrt{x-2}-1.$

(iii.) $\dfrac{2}{x-6+\sqrt{x}}+\dfrac{3}{\sqrt{x}-2}=\dfrac{4}{\sqrt{x}+3}.$

(iv.) $\sqrt[3]{x-a}-\sqrt[3]{x-b}=\sqrt[3]{b-a}.$

24. Find the condition that one root of $ax^2+bx+c=0$ shall be n times the other.

25. Find the value of $x^3-3x^2-8x+15$ when $x=3+i$.

26. Given $\log 648 = 2.81157$, $\log 864 = 2.93651$, find the logarithm of 3 and of 5.

27. Two trains run, without stopping, over the same 36 miles of rail. One of them travels 15 miles an hour faster than the other and accomplishes the distance in 12 minutes less. Find the speeds of the two trains.

28. Extract the square root of
$$9x^4-2x^3y+\tfrac{163}{9}x^2y^2-2xy^3+9y^4.$$

29. Find, by logarithms, the value of
$$\left\{\frac{15(.318)^{\frac{1}{7}}}{16}\right\}^{\frac{1}{11}}.$$

30. Simplify $\dfrac{1+ax^{-1}}{a^{-1}x^{-1}}\times\dfrac{a^{-1}-x^{-1}}{a^{-1}x-ax^{-1}}\div\dfrac{ax^{-1}}{x-a}.$

31. The men in a regiment can be arranged in a column twice as deep as its breadth; if the number be diminished by 206, the men can be arranged in a hollow square three deep having the same number of

men in each outer side of the square as there were in the depth of the column; how many men were there at first in the regiment?

32. Solve (i.) $2x^2 + xy + y^2 = 37,$
$8x^2 + 4xy + y^2 = 73.$

(ii.) $27x^3 + y^3 = 152,$
$3x^2y + xy^2 = 40.$

33. Simplify $8^{\frac{4}{3}} + \sqrt[3]{(2 \times 4^{-5})} - \sqrt[7]{2} \div 4^{-\frac{3}{7}} - (32)^{-\frac{3}{5}}.$

34. A man bought a field the length of which was to its breadth as 8 to 5. The number of dollars that he paid for 1 acre was equal to the number of rods in the length of the field; and 13 times the number of rods round the field equalled the number of dollars that it cost. Find the length and breadth of the field.

35. Solve (i.) $x^2 + xy + 3y^2 = 14 + 2\sqrt{2},$
$2x^2 + xy + 5y^2 = 24 + 2\sqrt{2}.$

(ii.) $2x + 3y = 10,$
$5x^2 + x + y = 4\frac{3}{4}.$

36. Find two numbers whose sum added to their product is 34, and the sum of whose squares diminished by their sum is 42.

37. Find the sixth term in the expansion of each of the following expressions:

(i.) $(a + 3b^{-2})^7.$ (ii.) $\left(2a - \dfrac{b^{\frac{1}{2}}}{c^{-2}}\right)^8.$ (iii.) $\left(\dfrac{x}{2} - \dfrac{\sqrt{y}}{3}\right)^9.$

38. A varies directly as B and inversely as C; $A = \frac{2}{3}$ when $B = \frac{3}{7}$ and $C = \frac{9}{14}$: find B when $A = \sqrt{48}$ and $C = \sqrt{75}$.

39. Solve (i.) $\sqrt{x + 12} + \sqrt[4]{x + 12} = 6.$

(ii.) $x^2 + y\sqrt{xy} = 9,$
$y^2 + x\sqrt{xy} = 18.$

40. Form an equation whose roots shall be the arithmetic and harmonic means between the roots of $x^2 - px + q = 0$.

z

CHAPTER XXXVII.

BINOMIAL THEOREM.

406. It may be shown by actual multiplication that

$$(a+b)(a+c)(a+d)(a+e)$$
$$= a^4 + (b+c+d+e)a^3 + (bc+bd+be+cd+ce+de)a^2$$
$$+ (bcd+bce+bde+cde)a + bcde \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

We may, however, write this result by inspection; for the complete product consists of the sum of a number of partial products each of which is formed by multiplying together four letters, *one* being taken from *each* of the four factors. If we examine the way in which the various partial products are formed, we see that

(1) The term a^4 is formed by taking the letter a out of *each* of the factors.

(2) The terms involving a^3 are formed by taking the letter a out of *any three* factors, in every way possible, and *one* of the letters b, c, d, e, out of the remaining factor.

(3) The terms involving a^2 are formed by taking the letter a out of *any two* factors, in every way possible, and *two* of the letters b, c, d, e, out of the remaining factors.

(4) The terms involving a are formed by taking the letter a out of *any one* factor, and *three* of the letters b, c, d, e, out of the remaining factors.

(5) The term independent of a is the product of all the letters b, c, d, e.

Ex. Find the value of $(a-2)(a+3)(a-5)(a+9)$.

The product

$$= a^4 + (-2+3-5+9)a^3 + (-6+10-18-15+27-45)a^2$$
$$+ (30-54+90-135)a + 270$$
$$= a^4 + 5a^3 - 47a^2 - 69a + 270.$$

407. If in equation (1) of the preceding article we suppose $c = d = e = b$, we obtain

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

We shall now employ the same method to prove a formula known as the **Binomial Theorem,** by which any binomial of the form $a + b$ can be raised to any assigned positive integral power.

408. To find the expansion of $(a + b)^n$ when n is a positive integer.

Consider the expression

$$(a+b)(a+c)(a+d)\cdots(a+k),$$

the number of factors being n.

The expansion of this expression is the continued product of the n factors, $a+b$, $a+c$, $a+d$, $\cdots a+k$, and every term in the expansion is of n dimensions, being a product formed by multiplying together n letters, *one* taken from each of these n factors.

The highest power of a is a^n, and is formed by taking the letter a from *each* of the n factors.

The terms involving a^{n-1} are formed by taking the letter a from *any* $n-1$ of the factors, and *one* of the letters $b, c, d, \cdots k$ from the remaining factor; thus the coefficient of a^{n-1} in the final product is the sum of the letters $b, c, d, \cdots k$; denote it by S_1.

The terms involving a^{n-2} are formed by taking the letter a from *any* $n-2$ of the factors, and *two* of the letters $b, c, d, \cdots k$ from the two remaining factors; thus the coefficient of a^{n-2} in the final product is the sum of the products of the letters $b, c, d, \cdots k$ taken two at a time; denote it by S_2.

And, generally, the terms involving a^{n-r} are formed by taking the letter a from *any* $n-r$ of the factors, and r of the letters $b, c, d, \cdots k$ from the r remaining factors; thus the coefficient of a^{n-r} in the final product is the sum of the products of the letters $b, c, d, \cdots k$ taken r at a time; denote it by S_r.

The last term in the product is $bcd \cdots k$; denote it by S_n.

Hence $\quad (a+b)(a+c)(a+d) \cdots (a+k)$

$$= a^n + S_1 a^{n-1} + S_2 a^{n-2} + \cdots + S_r a^{n-r} + \cdots + S_{n-1} a + S_n.$$

In S_1 the *number of terms* is n; in S_2 *the number of terms* is the same as the number of combinations of n things two at a time; that is, ${}^nC_2$; in S_3 *the number of terms* is ${}^nC_3$; and so on.

Now suppose $c, d, \cdots k$, each equal to b; then S_1 becomes ${}^nC_1 b$; S_2 becomes ${}^nC_2 b^2$; S_3 becomes ${}^nC_3 b^3$; and so on; thus

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + {}^nC_3 a^{n-3}b^3 + \cdots + {}^nC_n b^n);$$

substituting for ${}^nC_1$, ${}^nC_2$, $\cdots$ we obtain

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1 \cdot 2} a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3}b^3 + \cdots + b^n,$$

the series containing $n+1$ terms.

This is the *Binomial Theorem*, and the expression on the right side is said to be **the expansion** of $(a+b)^n$.

409. The coefficients in the expansion of $(a+b)^n$ are very conveniently expressed by the symbols ${}^nC_1, {}^nC_2, {}^nC_3 \ldots {}^nC_n$. We shall, however, sometimes further abbreviate them by omitting n, and writing $C_1, C_2, C_3, \ldots C_n$. With this notation we have

$$(a+b)^n = a^n + C_1 a^{n-1}b + C_2 a^{n-2}b^2 + C_3 a^{n-3}b^3 + \cdots + C_n b^n.$$

If we write $-b$ in the place of b, we obtain

$$(a-b)^n = a^n + C_1 a^{n-1}(-b) + C_2 a^{n-2}(-b)^2 + C_3 a^{n-3}(-b)^3 + \cdots + C_n(-b)^n$$
$$= a^n - C_1 a^{n-1}b + C_2 a^{n-2}b^2 - C_3 a^{n-3}b^3 + \cdots + (-1)^n C_n b^n.$$

Thus the terms in the expansion of $(a+b)^n$ and $(a-b)^n$ are *numerically* the same, but in $(a-b)^n$ they are alternately positive and negative, and the last term is positive or negative according as n is even or odd.

Ex. 1. Find the expansion of $(a+y)^6$.

By the formula, the expansion

$$= a^6 + {}^6C_1a^5y + {}^6C_2a^4y^2 + {}^6C_3a^3y^3 + {}^6C_4a^2y^4 + {}^6C_5ay^5 + {}^6C_6y^6$$
$$= a^6 + 6\,a^5y + 15\,a^4y^2 + 20\,a^3y^3 + 15\,a^2y^4 + 6\,ay^5 + y^6,$$

on calculating the values of ${}^6C_1$, ${}^6C_2$, ${}^6C_3$, ….

Ex. 2. Find the expansion of $(a-2x)^7$.

$$(a-2x)^7 = a^7 - {}^7C_1a^6(2x) + {}^7C_2a^5(2x)^2 - {}^7C_3a^4(2x)^3 + \cdots \text{ to 8 terms.}$$

Now remembering that ${}^nC_r = {}^nC_{n-r}$ after calculating the coefficients up to ${}^7C_3$, the rest may be written down at once; for ${}^7C_4 = {}^7C_3$; ${}^7C_5 = {}^7C_2$; and so on. Hence

$$\begin{aligned}(a-2x)^7 &= a^7 - 7\,a^6(2x) + \frac{7\cdot 6}{1\cdot 2}\,a^5(2x)^2 - \frac{7\cdot 6\cdot 5}{1\cdot 2\cdot 3}\,a^4(2x)^3 + \cdots \\ &= a^7 - 7\,a^6(2x) + 21\,a^5(2x)^2 - 35\,a^4(2x)^3 + 35\,a^3(2x)^4 \\ &\qquad - 21\,a^2(2x)^5 + 7\,a(2x)^6 - (2x)^7 \\ &= a^7 - 14\,a^6x + 84\,a^5x^2 - 280\,a^4x^3 + 560\,a^3x^4 \\ &\qquad - 672\,a^2x^5 + 448\,ax^6 - 128\,x^7.\end{aligned}$$

410. The $(r+1)$th or General Term. In the expansion of $(a+b)^n$, the coefficient of the second term is ${}^nC_1$; of the third term is ${}^nC_2$; of the fourth term is ${}^nC_3$; and so on; the suffix in each term being one less than the number of the term to which it applies; hence ${}^nC_r$ is the coefficient of the $(r+1)$th term. This is called the *general term*, because by giving to r different numerical values any of the coefficients may be found from ${}^nC_r$; and by giving to a and b their appropriate indices any assigned term may be obtained. Thus the $(r+1)$th term may be written

$${}^nC_ra^{n-r}b^r, \text{ or } \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}\,^*}\,a^{n-r}b^r.$$

In applying this formula to any particular case, it should be observed that *the index of b is the same as the suffix of C, and that the sum of the indices of a and b is n.*

* See Art. 392, Cor.

Ex. 1. Find the fifth term of $(a + 2x^3)^{17}$.

Here $(r + 1) = 5$, therefore

$$\text{the required term} = {}^{17}C_4 a^{13}(2x^3)^4$$
$$= \frac{17 \cdot 16 \cdot 15 \cdot 14}{1 \cdot 2 \cdot 3 \cdot 4} \times 16 a^{13}x^{12}$$
$$= 38080\, a^{13}x^{12}.$$

Ex. 2. Find the fourteenth term of $(3 - a)^{15}$.

Here $r + 1 = 14$, therefore

$$\text{the required term} = {}^{15}C_{13}(3)^2(-a)^{13}$$
$$= {}^{15}C_2 \times (-9a^{13}) \qquad \text{[Art. 395.]}$$
$$= -945\, a^{13}.$$

411. Simplest Form of the Binomial Theorem. The most convenient form of the binomial theorem is the expansion of $(1 + x)^n$. This is obtained from the general formula of Art. 408, by writing 1 in the place of a, and x in the place of b. Thus

$$(1 + x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_r x^r + \cdots + {}^nC_n x^n$$
$$= 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \cdots + x^n,$$

the *general term* being $\dfrac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r} x^r$.

412. The expansion of a binomial may always be made to depend upon the case in which the first term is unity; thus

$$(a + b)^n = \left\{ a\left(1 + \frac{b}{a}\right)\right\}^n = a^n(1 + c)^n, \text{ where } c = \frac{b}{a}.$$

Ex. Find the coefficient of x^{16} in the expansion of $(x^2 - 2x)^{10}$.

We have $$(x^2 - 2x)^{10} = x^{20}\left(1 - \frac{2}{x}\right)^{10};$$

and, since x^{20} multiplies every term in the expansion of $\left(1 - \dfrac{2}{x}\right)^{10}$, we have in this expansion to seek the coefficient of the term which contains $\dfrac{1}{x^4}$.

Hence the required coefficient $= {}^{10}C_4(-2)^4$

$$= \frac{10 \cdot 9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} \times 16 = 3360.$$

PROOF BY MATHEMATICAL INDUCTION.

413. By actual multiplication we obtain the following identities:

$$(a+b)^2 = a^2 + 2ab + b^2,$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$
$$(a+b)^4 = a \ + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$$

Selecting any one of these, and rewriting so as to exhibit the laws of formation of exponents and coefficients, we have

$$(a+b)^4 = a^4 + \frac{4}{1}a^{4-1}b + \frac{4\cdot 3}{1\cdot 2}a^{4-2}b^2 + \frac{4\cdot 3\cdot 2}{1\cdot 2\cdot 3}a^{4-3}b^3 + \frac{4\cdot 3\cdot 2\cdot 1}{1\cdot 2\cdot 3\cdot 4}a^0b^4 \text{ (Art. 216).}$$

If these laws of formation hold for $(a+b)^n$, n being any positive integer, then

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots \quad . \quad . \quad . \quad (1).$$

Multiplying each side of the assumed identity by $(a+b)$ and combining terms, we obtain

$$(a+b)^{n+1} = a^{n+1} + (n+1)a^nb + \frac{n(n+1)}{1\cdot 2}a^{n-1}b^2 + \frac{n(n+1)(n-1)}{1\cdot 2\cdot 3}a^{n-2}b^3 + \cdots \quad . \quad . \quad (2).$$

It will be seen that n in (1) is, in every instance, replaced by $(n+1)$ in (2). Hence if the theorem be true for any value of n, it will be true for the next higher value. We have shown by multiplication that the theorem is true when n successively equals 2, 3, and 4; hence it is true when $n = 5$, and so on indefinitely. The theorem is therefore true for *all* positive integral values of n.

414. In the expansion

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \cdots$$

we observe that in *any* term

(1) The exponent of b, the second term of the binomial, is *one less* than the number of the term from the first.

(2) The sum of the exponents is n.

(3) The *last* factor of the denominator of the coefficient is the same as the exponent of the *second term* of the binomial.

(4) The *last* factor of the numerator of the coefficient is the exponent of the *first term* of the binomial increased by 1.

Hence the $(r+1)$th or **general term** of $(a+b)^n$ is

$$\frac{n(n-1)\cdots(n-r+1)}{1\cdot 2\cdot 3\cdots r}a^{n-r}b^r.$$

Ex. Find the 6th term in the expansion of $(2a+b)^{10}$.

Here $n=10$, and $r+1=6$.

We have $\dfrac{10\cdot 9\cdot 8\cdot 7\cdot 6}{1\cdot 2\cdot 3\cdot 4\cdot 5}(2a)^5b^5 = \dfrac{3\cdot 2\cdot 7\cdot 6}{1}(2a)^5b^5$

$$= 252(2)^5a^5b^5 = 8064\,a^5b^5.$$

Note. The student should observe that the coefficient contains the *same number* of factors in both numerator and denominator.

EXAMPLES XXXVII. a.

Expand the following binomials;

1. $(x+2)^4$.
2. $(x+3)^5$.
3. $(a+x)^7$.
4. $(a-x)^5$.
5. $(1-2y)^5$.
6. $\left(a-\frac{3}{b}\right)^7$.
7. $\left(2-\frac{x}{2}\right)^6$.
8. $\left(2x+\frac{y}{2}\right)^4$.
9. $\left(ax+\frac{y}{a}\right)^9$.

Write in simplest form:

10. The 4th term of $(1+x)^{12}$.
11. The 6th term of $(2-y)^8$.
12. The 5th term of $(a-5b)^7$.
13. The 15th term of $(2x-1)^{17}$.
14. The 7th term of $\left(1-\frac{1}{x}\right)^{10}$.
15. The 6th term of $\left(3x+\frac{a}{2}\right)^9$.

16. Find the value of $(x-\sqrt{3})^4+(x+\sqrt{3})^4$.

17. Expand $(\sqrt{1-x^2}+1)^5-(\sqrt{1-x^2}-1)^5$.

18. Find the coefficient of x^{12} in $(x^2+2x)^{10}$.

19. Find the coefficient of x in $\left(x^2-\dfrac{a}{2x}\right)^{14}$.

20. Find the term independent of x in $\left(2x^2-\dfrac{1}{x}\right)^{12}$.

21. Find the coefficient of x^{-20} in $\left(\dfrac{x^2}{3}-\dfrac{2}{x^3}\right)^{15}$.

415. Equal Coefficients. *In the expansion of* $(1+x)^n$ *the coefficients of terms equidistant from the beginning and end are equal.*

The coefficient of the $(r+1)$th term from the beginning is ${}^nC_r$.

The $(r+1)$th term from the end has $n+1-(r+1)$, or $n-r$ terms before it; therefore counting from the beginning it is the $(n-r+1)$th term, and its coefficient is ${}^nC_{n-r}$, which has been shown to be equal to ${}^nC_r$ [Art. 395]. Hence the proposition follows.

416. Greatest Coefficient. *To find the greatest coefficient in the expansion of* $(1+x)^n$.

The coefficient of the general term of $(1+x)^n$ is ${}^nC_r$; and we have only to find for what value of r this is greatest.

By Art. 401, when n is even, the greatest coefficient is ${}^nC_{\frac{n}{2}}$; when n is odd, it is ${}^nC_{\frac{n-1}{2}}$, or ${}^nC_{\frac{n+1}{2}}$; these coefficients being equal.

417. Greatest Term. *To find the greatest term in the expansion of* $(a+b)^n$.

We have $$(a+b)^n=a^n\left(1+\frac{b}{a}\right)^n;$$

therefore, since a^n multiplies every term in $\left(1+\dfrac{b}{a}\right)^n$, it will be sufficient to find the greatest term in this latter expansion.

Let the rth and $(r+1)$th be any two consecutive terms. The $(r+1)$th term is obtained by multiplying the rth term by $\frac{n-r+1}{r}\cdot\frac{b}{a}$; that is, by $\left(\frac{n+1}{r}-1\right)\frac{b}{a}$. [Art. 410.]

The factor $\frac{n+1}{r}-1$ decreases as r increases; hence the $(r+1)$th term is not always greater than the rth term, but only until $\left(\frac{n+1}{r}-1\right)\frac{b}{a}$ becomes equal to 1, or less than 1.

Now $\left(\frac{n+1}{r}-1\right)\frac{b}{a}>1$, so long as $\frac{n+1}{r}-1>\frac{a}{b}$;

that is, $$\frac{n+1}{r}>\frac{a}{b}+1, \text{ or } \frac{(n+1)b}{a+b}>r \quad . \quad . \quad . \quad (1).$$

If $\frac{(n+1)b}{a+b}$ be an integer, denote it by p; then if $r=p$ the multiplying factor becomes 1, and the $(p+1)$th term is equal to the pth; and these are greater than any other term.

If $\frac{(n+1)b}{a+b}$ be not an integer, denote its integral part by q; then the greatest value of r consistent with (1) is q; hence the $(q+1)$th term is the greatest.

Since we are only concerned with the *numerically greatest term*, the investigation will be the same for $(a-b)^n$; therefore in any numerical example it is unnecessary to consider the sign of the second term of the binomial. Also it will be found best to work each example independently of the general formula.

Ex. Find the greatest term in the expansion of $(1+4x)^8$, when x has the value $\frac{1}{3}$.

Denote the rth and $(r+1)$th terms by T_r and T_{r+1} respectively; then

$$T_{r+1}=\frac{8-r+1}{r}\cdot 4x\times T_r=\frac{9-r}{r}\times\frac{4}{3}\times T_r;$$

hence $$T_{r+1}>T_r, \text{ so long as } \frac{9-r}{r}\times\frac{4}{3}>1;$$

that is, $$36-4r>3r, \text{ or } 36>7r.$$

The greatest value of r consistent with this is 5; hence the greatest term is the sixth, and its value

$$= {}^8C_5 \times (\tfrac{4}{3})^5 = {}^8C_3 \times (\tfrac{4}{3})^5 = \tfrac{57344}{243}.$$

418. Sum of the Coefficients. *To find the sum of the coefficients in the expansion of* $(1+x)^n$.

In the identity

$$(1+x)^n = 1 + C_1x + C_2x^2 + C_3x^3 + \cdots + C_nx^n;$$

put $x = 1$; thus

$$2^n = 1 + C_1 + C_2 + C_3 + \cdots + C_n$$
$$= \text{sum of the coefficients.}$$

COR. $\qquad C_1 + C_2 + C_3 + \cdots + C_n = 2^n - 1;$

that is, the total number of combinations of n things *taking some or all of them at a time* is $2^n - 1$. [See Art. 400.]

419. Sums of Coefficients equal. *To prove that in the expansion of* $(1+x)^n$, *the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms.*

In the identity

$$(1+x)^n = 1 + C_1x + C_2x^2 + C_3x^3 + \cdots + C_nx^n,$$

put $x = -1$; thus

$$0 = 1 - C_1 + C_2 - C_3 + C_4 - C_5 + \cdots;$$
$$\therefore 1 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots.$$

420. Expansion of Multinomials. The Binomial Theorem may also be applied to expand expressions which contain more than two terms.

Ex. Find the expansion of $(x^2 + 2x - 1)^3$.

Regarding $2x - 1$ as a single term, the expansion

$$= (x^2)^3 + 3(x^2)^2(2x-1) + 3x^2(2x-1)^2 + (2x-1)^3$$
$$= x^6 + 6x^5 + 9x^4 - 4x^3 - 9x^2 + 6x - 1, \text{ on reduction.}$$

421. Binomial Theorem for Negative or Fractional Index. For a full discussion of the Binomial Theorem when the index is not restricted to positive integral values the student is referred to Chapter XLV. It is there shown that when x is less than unity, the formula

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1\cdot 2}x^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$

is true for any value of n.

When n is negative or fractional the number of terms in the expansion is unlimited, but in any particular case we may write down as many terms as we please, or we may find the coefficient of any assigned term.

Ex. 1. Expand $(1+x)^{-3}$ to four terms.

$$(1+x)^{-3} = 1 + (-3)x + \frac{(-3)(-3-1)}{1\cdot 2}x^2 + \frac{(-3)(-3-1)(-3-2)}{1\cdot 2\cdot 3}x^3 + \cdots$$
$$= 1 - 3x + \frac{3\cdot 4}{1\cdot 2}x^2 - \frac{3\cdot 4\cdot 5}{1\cdot 2\cdot 3}x^3 + \cdots$$
$$= 1 - 3x + 6x^2 - 10x^3 + \cdots$$

Ex. 2. Expand $(4+3x)^{\frac{3}{2}}$ to four terms.

$$(4+3x)^{\frac{3}{2}} = 4^{\frac{3}{2}}\left(1+\frac{3x}{4}\right)^{\frac{3}{2}} = 8\left(1+\frac{3x}{4}\right)^{\frac{3}{2}}$$
$$= 8\left[1+\frac{3}{2}\cdot\frac{3x}{4} + \frac{\frac{3}{2}(\frac{3}{2}-1)}{1\cdot 2}\left(\frac{3x}{4}\right)^2 + \frac{\frac{3}{2}(\frac{3}{2}-1)(\frac{3}{2}-2)}{1\cdot 2\cdot 3}\left(\frac{3x}{4}\right)^3 + \cdots\right]$$
$$= 8\left[1+\frac{3}{2}\cdot\frac{3x}{4} + \frac{3}{8}\cdot\frac{9x^2}{16} - \frac{1}{16}\cdot\frac{27x^3}{64} + \cdots\right]$$
$$= 8 + 9x + \tfrac{27}{16}x^2 - \tfrac{27}{128}x^3 + \cdots.$$

422. In finding the **general term** we must now use the formula

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r}x^r$$

written in full; for the symbol ${}^nC_r$ cannot be employed when n is fractional or negative.

Ex. 1. Find the general term in the expansion of $(1+x)^{\frac{1}{2}}$.

$$\text{The } (r+1)\text{th term} = \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-r+1)}{\lfloor r}x^r$$
$$= \frac{1(-1)(-3)(-5)\cdots(-2r+3)}{2^r\lfloor r}x^r.$$

The number of factors in the numerator is r, and $r-1$ of these are negative; therefore, by taking -1 out of each of these negative factors, we may write the above expression

$$(-1)^{r-1}\frac{1\cdot 3\cdot 5\cdots(2r-3)}{2^r\lfloor r}x^r.$$

Ex. 2. Find the general term in the expansion of $(1-x)^{-3}$.

$$\begin{aligned}\text{The } (r+1)\text{th term} &= \frac{(-3)(-4)(-5)\cdots(-3-r+1)}{\lfloor r}(-x)^r\\ &= (-1)^r\frac{3\cdot 4\cdot 5\cdots(r+2)}{\lfloor r}(-1)^r x^r\\ &= (-1)^{2r}\frac{3\cdot 4\cdot 5\cdots(r+2)}{1\cdot 2\cdot 3\cdots r}x^r\\ &= \frac{(r+1)(r+2)}{1\cdot 2}x^r,\end{aligned}$$

by removing like factors from the numerator and denominator.

423. The following expansions should be remembered:

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots + x^r + \cdots.$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots + (r+1)x^r + \cdots$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3\cdots + \frac{(r+1)(r+2)}{1\cdot 2}x^r + \cdots.$$

424. The following example illustrates a useful application of the Binomial Theorem.

Ex. Find the cube root of 126 to 5 places of decimals.

$$\begin{aligned}(126)^{\frac{1}{3}} = (5^3+1)^{\frac{1}{3}} &= 5\left(1+\frac{1}{5^3}\right)^{\frac{1}{3}}\\ &= 5\left(1+\frac{1}{3}\cdot\frac{1}{5^3}-\frac{1}{9}\cdot\frac{1}{5^6}+\frac{5}{81}\cdot\frac{1}{5^9}-\cdots\right)\\ &= 5+\frac{1}{3}\cdot\frac{1}{5^2}-\frac{1}{9}\cdot\frac{1}{5^5}+\frac{1}{81}\cdot\frac{1}{5^7}-\cdots\\ &= 5+\frac{1}{3}\cdot\frac{2^2}{10^2}-\frac{1}{9}\cdot\frac{2^5}{10^5}+\frac{1}{81}\cdot\frac{2^7}{10^7}-\cdots\\ &= 5+\frac{.04}{3}-\frac{.00032}{9}+\frac{.0000128}{81}-\cdots\\ &= 5+.013333\cdots-.000035\cdots+\cdots\\ &= 5.01329, \text{ to five places of decimals.}\end{aligned}$$

EXAMPLES XXXVII. b.

In the following expansions find which is the greatest term:

1. $(x+y)^{17}$ when $x=4$, $y=3$.
2. $(x-y)^{28}$ when $x=9$, $y=4$.
3. $(1+x)^4$ when $x=\frac{2}{3}$.
4. $(a-4b)^{15}$ when $a=12$, $b=2$.
5. $(7x+2y)^{30}$ when $x=8$, $y=14$.
6. $(2x+3)^n$ when $x=\frac{5}{2}$, $n=15$.

7. In the expansion of $(1+x)^{25}$ the coefficients of the $(2r+1)$th and $(r+5)$th terms are equal: find r.

8. Find n when the coefficients of the 16th and 26th terms of $(1+x)^n$ are equal.

9. Find the relation between r and n in order that the coefficients of $(r+3)$th and $(2r-3)$th terms of $(1+x)^{3n}$ may be equal.

10. Find the coefficient of x^m in the expansion of $\left(x^2+\frac{1}{x}\right)^{2m}$.

11. Find the middle term of $(1+x)^{2n}$ in its simplest form.

12. Find the sum of the coefficients of $(x+y)^{16}$.

13. Find the sum of the coefficients of $(3x+y)^9$.

14. Find the rth term from the beginning and the rth term from the end of $(a+2x)^n$.

15. Expand $(a^2+2a+1)^3$ and $(x^2-4x+2)^3$.

Expand to four terms the following expressions:

16. $(1+x)^{\frac{1}{3}}$.
17. $(1+x)^{\frac{3}{4}}$.
18. $(1+x)^{\frac{2}{5}}$.
19. $(1+3x)^{-2}$.
20. $(1-x^2)^{-3}$.
21. $(1+3x)^{-4}$.
22. $(2+x)^{-3}$.
23. $(1+2x)^{-\frac{1}{2}}$.
24. $(a-2x)^{-\frac{3}{2}}$.

Write in simplest form:

25. The 5th term and the 10th term of $(1+x)^{-\frac{3}{2}}$.
26. The 3d term and the 11th term of $(1+2x)^{\frac{11}{2}}$.
27. The 4th term and the $(r+1)$th term of $(1+x)^{-2}$.
28. The 7th term and the $(r+1)$th term of $(1-x)^{\frac{1}{2}}$.
29. The $(r+1)$th term of $(a-bx)^{-1}$, and of $(1-nx)^{\frac{1}{n}}$.

Find to four places of decimals the value of

30. $\sqrt[3]{122}$.
31. $\sqrt[4]{620}$.
32. $\sqrt[5]{31}$.
33. $1 \div \sqrt{99}$.

Find the value of

34. $(x+\sqrt{2})^4+(x-\sqrt{2})^4$.

35. $(\sqrt{x^2-a^2}+x)^5-(\sqrt{x^2-a^2}-x)^5$.

36. $(\sqrt{2}+1)^6-(\sqrt{2}-1)^6$.

37. $(2-\sqrt{1-x})^6+(2+\sqrt{1-x})^6$.

38. Find the middle term of $\left(\frac{a}{x}+\frac{x}{a}\right)^{10}$.

39. Find the middle term of $\left(1-\frac{x^2}{2}\right)^{14}$.

40. Find the coefficient of x^{18} in $\left(x^2+\frac{3a}{x}\right)^{15}$.

41. Find the coefficient of x^{18} in $(ax^4-bx)^9$.

42. Find the coefficients of x^{32} and x^{-17} in $\left(x^4-\frac{1}{x^3}\right)^{15}$.

43. Find the two middle terms of $\left(3a-\frac{a^3}{6}\right)^9$.

Write in simplest form

44. The 8th term of $(1+2x)^{-\frac{1}{2}}$.

45. The 11th term of $(1-2x^3)^{\frac{11}{2}}$.

46. The 10th term of $(1+3a^2)^{\frac{16}{3}}$.

47. The 5th term of $(3a-2b)^{-1}$.

48. The $(r+1)$th term of $(1-x)^{-2}$.

49. The $(r+1)$th term of $(1-x)^{-4}$.

50. The $(r+1)$th term of $(1+x)^{\frac{1}{2}}$.

51. The $(r+1)$th term of $(1+x)^{\frac{11}{3}}$.

52. The 14th term of $(2^{10}-2^7x)^{\frac{13}{2}}$.

53. The 7th term of $(3^8+6^4x)^{\frac{11}{4}}$.

CHAPTER XXXVIII.

LOGARITHMS.

425. DEFINITION. The **logarithm** of any number to a given **base** is the index of the power to which the base must be raised in order to equal the given number. Thus if $a^x = N$, x is called the logarithm of N to the base a.

EXAMPLES. (1) Since $3^4 = 81$, the logarithm of 81 to base 3 is 4.

(2) Since $10^1 = 10,\ 10^2 = 100,\ 10^3 = 1000, \cdots$

the natural numbers 1, 2, 3, $\cdots$ are respectively the logarithms of 10, 100, 1000, $\cdots$ to base 10.

426. The logarithm of N to base a is usually written $\log_a N$, so that the same meaning is expressed by the two equations

$$a^x = N; \ x = \log_a N.$$

Ex. Find the logarithm of $32\sqrt[5]{4}$ to base $2\sqrt{2}$.

Let x be the required logarithm; then, by definition,

$$(2\sqrt{2})^x = 32\sqrt[5]{4};$$

$$\therefore\ (2 \cdot 2^{\frac{1}{2}})^x = 2^5 \cdot 2^{\frac{2}{5}};$$

$$\therefore\ 2^{\frac{3}{2}x} = 2^{5+\frac{2}{5}};$$

hence, by equating the indices, $\frac{3}{2}x = \frac{27}{5}$;

$$\therefore\ x = \frac{18}{5} = 3.6.$$

427. When it is understood that a particular system of logarithms is in use, the suffix denoting the base is omitted. Thus in arithmetical calculations in which 10 is the base, we usually write $\log 2, \log 3, \cdots$ instead of $\log_{10} 2, \log_{10} 3, \cdots$.

Logarithms to the base 10 are known as **Common Logarithms**; this system was first introduced in 1615 by Briggs, a contemporary of Napier the inventor of Logarithms.

PROPERTIES OF LOGARITHMS.

428. Logarithm of Unity. *The logarithm of* 1 *is* 0.

For $a^0 = 1$ for all values of a; therefore $\log 1 = 0$, whatever the base may be.

429. Logarithm of the Base. *The logarithm of the base itself is* 1.

For $a^1 = a$; therefore $\log_a a = 1$.

430. Logarithm of Zero. *The logarithm of* 0, *in any system whose base is greater than unity, is minus infinity.*

For $$a^{-\infty} = \frac{1}{a^\infty} = 0.$$

Also, since $a^{+\infty} = \infty$, *the logarithm of* $+\infty$ *is* $+\infty$.

431. Logarithm of a Product. *The logarithm of a product is the sum of the logarithms of its factors.*

Let MN be the product; let a be the base of the system, and suppose

$$x = \log_a M, \quad y = \log_a N;$$

so that $$a^x = M, \qquad a^y = N.$$

Thus the product $MN = a^x \times a^y = a^{x+y}$;

whence, by definition, $$\log_a MN = x + y = \log_a M + \log_a N.$$

Similarly, $$\log_a MNP = \log_a M + \log_a N + \log_a P;$$

and so on for any number of factors.

Ex. $\log 42 = \log (2 \times 3 \times 7) = \log 2 + \log 3 + \log 7.$

432. Logarithm of a Quotient. *The logarithm of a quotient is the logarithm of the dividend minus the logarithm of the divisor.*

Let $\frac{M}{N}$ be the fraction, and suppose

$$x = \log_a M, \quad y = \log_a N;$$

so that $$a^x = M, \qquad a^y = N.$$

Thus the fraction $$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y};$$

whence, by definition, $\log_a \frac{M}{N} = x - y = \log_a M - \log_a N.$

Ex. $\log (2\frac{1}{7}) = \log \frac{15}{7} = \log 15 - \log 7$
$= \log (3 \times 5) - \log 7 = \log 3 + \log 5 - \log 7.$

433. Logarithm of a Power. *The logarithm of a number raised to any power, integral or fractional, is the logarithm of the number multiplied by the index of the power.*

Let $\log_a (M^p)$ be required, and suppose

$$x = \log_a M, \text{ so that } a^x = M;$$

then $$M^p = (a^x)^p = a^{px};$$

whence, by definition, $\log_a (M^p) = px;$

that is, $$\log_a (M^p) = p \log_a M.$$

Similarly, $$\log_a (M^{\frac{1}{r}}) = \frac{1}{r} \log^a M.$$

Ex. Express the logarithm of $\frac{\sqrt{a^3}}{c^5 b^2}$ in terms of $\log a$, $\log b$, and $\log c$.

$$\log \frac{\sqrt{a^3}}{c^5b^2} = \log \frac{a^{\frac{3}{2}}}{c^5b^2} = \log a^{\frac{3}{2}} - \log (c^5b^2)$$
$$= \tfrac{3}{2} \log a - (\log c^5 + \log b^2) = \tfrac{3}{2} \log a - 5 \log c - 2 \log b.$$

434. From the equation $10^x = N$, it is evident that **common logarithms** will not in general be integral, and that they will not always be positive.

For instance, $3154 > 10^3$ and $< 10^4$;

$\therefore \log 3154 = 3 +$ a fraction.

Again, $.06 > 10^{-2}$ and $< 10^{-1}$;

$\therefore \log .06 = -2 +$ a fraction.

Negative numbers have no common logarithms.

435. Definition. The integral part of a logarithm is called the **characteristic**, and the decimal part, when it is so written that it is positive, is called the **mantissa**.

The characteristic of the logarithm of any number to the **base 10** can be written by inspection, as we shall now show.

436. The Characteristic of the Logarithm of Any Number Greater than Unity. It is clear that a number with two digits in its integral part lies between 10^1 and 10^2; a number with three digits in its integral part lies between 10^2 and 10^3; and so on. Hence a number with n digits in its integral part lies between 10^{n-1} and 10^n.

Let N be a number whose integral part contains n digits; then

$$N = 10^{(n-1) + \text{a fraction}};$$

$$\therefore \log N = (n - 1) + \text{a fraction}.$$

Hence the characteristic is $n - 1$; that is, *the characteristic of the logarithm of a number greater than unity is less by one than the number of digits in its integral part, and is positive.*

437. The Characteristic of the Logarithm of a Decimal Fraction. A decimal with one cipher immediately after the decimal point, such as .0324, being greater than .01 and less than .1, lies between 10^{-2} and 10^{-1}; a number with two ciphers after the decimal point lies between 10^{-3} and 10^{-2}; and so on. Hence a decimal fraction with n ciphers immediately after the decimal point lies between $10^{-(n+1)}$ and 10^{-n}.

Let D be a decimal beginning with n ciphers; then

$$D = 10^{-(n+1) + \text{a fraction}};$$

$$\therefore \log D = -(n + 1) + \text{a fraction}.$$

Hence the characteristic is $-(n + 1)$; that is, *the characteristic of the logarithm of a decimal fraction is greater by unity than the number of ciphers immediately after the decimal point and is negative.*

438. Advantages of Common Logarithms. Common logarithms, because of the two great advantages of the base 10, are in common use. These two advantages are as follows:

(1) From the results already proved it is evident that the characteristics can be written by inspection, so that only the mantissæ have to be registered in the Tables.

(2) The mantissæ are the same for the logarithms of all numbers which have *the same significant digits;* so that it is sufficient to tabulate the mantissæ of the logarithms of *integers.*

This proposition we proceed to prove.

439. Let N be any number, then since multiplying or dividing by a power of 10 merely alters the position of the decimal point without changing the sequence of figures, it follows that $N \times 10^p$, and $N \div 10^q$, where p and q are any integers, are numbers whose significant digits are the same as those of N.

Now $\log(N \times 10^p) = \log N + p \log 10 = \log N + p \quad . \ (1).$

Again, $\log(N \div 10^q) = \log N - q \log 10 = \log N - q \quad . \ (2).$

In (1) an integer is added to $\log N$, and in (2) an integer is subtracted from $\log N$; that is, the *mantissa or decimal portion of the logarithm remains unaltered.*

In this and the three preceding articles the mantissæ have been supposed positive. In order to secure the advantages of Briggs' system, we arrange our work so as *always to keep the mantissa positive,* so that when the mantissa of any logarithm has been taken from the Tables the characteristic is prefixed with its appropriate sign, according to the rules already given.

440. In the case of a negative logarithm the minus sign is written *over the characteristic,* and not before it, to indicate that the characteristic alone is negative, and not the whole expression. Thus $\bar{4}.30103$, the logarithm of .0002, is equivalent to $-4 + .30103$, and must be distinguished from -4.30103, an expression in which both the integer and the decimal are negative. In working with negative logarithms an arithmetical artifice will sometimes be necessary in order to make the mantissa positive. For instance, a result such as -3.69897, in which the whole expression is negative, may be transformed by subtracting 1 from the characteristic and adding 1 to the mantissa. Thus,

$$-3.69897 = -4 + (1 - .69897) = \bar{4}.30103.$$

Ex. 1. Required the logarithm of .0002432.

In Seven-Place Tables we find that 3859636 is the mantissa of log 2432 (the decimal point as well as the characteristic being omitted); and, by Art. 437, the characteristic of the logarithm of the given number is -4;

$$\therefore \log .0002432 = \bar{4}.3859636.$$

This may be written $6.3859636 - 10$.

Ex. 2. Find the value of $\sqrt[5]{.00000165}$, given

$$\log 165 = 2.2174839, \quad \log 697424 = 5.8434968.$$

Let x denote the value required; then

$$\log x = \log (.00000165)^{\frac{1}{5}} = \tfrac{1}{5}\log (.00000165) = \tfrac{1}{5}(\bar{6}.2174839);$$

the *mantissa* of log .00000165 being the same as that of log 165, and the *characteristic* being prefixed by the rule.

Now $\quad \tfrac{1}{5}(\bar{6}.2174839) = \tfrac{1}{5}(\overline{10} + 4.2174839) = \bar{2}.8434968$

and .8434968 is the mantissa of log 697424; hence x is a number consisting of these same digits, but with one cipher after the decimal point. [Art. 437.]

Thus $$x = .0697424.$$

441. Logarithms transformed from Base *a* to Base *b*. Suppose that the logarithms of all numbers to base a are known and tabulated.

Let N be any number whose logarithm to base b is required.

Let $y = \log_b N$, so that $b^y = N$;

$$\therefore \log_a (b^y) = \log_a N;$$

that is, $$y \log_a b = \log_a N;$$

$$\therefore y = \frac{1}{\log_a b} \times \log_a N,$$

or $$\log_b N = \frac{1}{\log_a b} \times \log_a N \quad \ldots \ldots \quad (1).$$

Now since N and b are given, $\log_a N$ and $\log_a b$ are known from the Tables, and thus $\log_b N$ may be found.

Hence *to transform logarithms from base a to base b we multiply them all by* $\dfrac{1}{\log_a b}$; this is a constant quantity, and is given by the Tables; it is known as the *modulus*.

Cor. If in equation (1) we put a for N, we obtain

$$\log_b a = \frac{1}{\log_a b} \times \log_a a = \frac{1}{\log_a b};$$

$$\therefore\ \log_b a \times \log_a b = 1.$$

442. Logarithms in Arithmetical Calculation. The following examples illustrate the utility of logarithms in facilitating arithmetical calculation.

Ex. 1. Given $\log 3 = .4771213$, find $\log\{(2.7)^3 \times (.81)^{\frac{4}{5}} \div (90)^{\frac{5}{4}}\}$.

$$\begin{aligned}\text{The required value} &= 3 \log \tfrac{27}{10} + \tfrac{4}{5} \log \tfrac{81}{100} - \tfrac{5}{4} \log 90 \\ &= 3(\log 3^3 - 1) + \tfrac{4}{5}(\log 3^4 - 2) - \tfrac{5}{4}(\log 3^2 + 1) \\ &= (9 + \tfrac{16}{5} - \tfrac{5}{2}) \log 3 - (3 + \tfrac{8}{5} + \tfrac{5}{4}) \\ &= \tfrac{97}{10} \log 3 - 5\tfrac{17}{20} = 4.6280766 - 5.85 = \bar{2}.7780766.\end{aligned}$$

The student should notice that the logarithm of 5 and its powers can always be obtained from $\log 2$; thus

$$\log 5 = \log \tfrac{10}{2} = \log 10 - \log 2 = 1 - \log 2.$$

Ex. 2. Find the number of digits in 875^{16}, given

$$\log 2 = .3010300, \quad \log 7 = .8450980.$$

$$\begin{aligned}\log (875^{16}) &= 16 \log (7 \times 125) = 16 (\log 7 + 3 \log 5) \\ &= 16 (\log 7 + 3 - 3 \log 2) \\ &= 16 \times 2.9420080 = 47.072128;\end{aligned}$$

hence the number of digits is 48. [Art. 436.]

EXAMPLES XXXVIII. a.

1. Find the logarithms of $\sqrt{32}$ and .03125 to base $\sqrt[3]{2}$, and 100 and .00001 to base .01.

2. Find the value of $\log_4 512$, $\log_5 .0016$, $\log_{81} \frac{1}{27}$, $\log_{49} 343$.

3. Write the numbers whose logarithms to bases 25, $\dot{3}$, .02, 1, -4, 1.7, 1000, are $\frac{1}{2}$, -2, -3, 5, -1, 2, $-\frac{2}{3}$ respectively.

Simplify the expressions:

4. $\log \dfrac{(ab^2c^4)^{\frac{1}{6}}}{\sqrt[9]{a^{-3}b^3c^6}}$.

5. $\log \left\{ \left(\dfrac{x^4y^{-3}}{x^{-1}y^2}\right)^{-3} \div \left(\dfrac{x^{-2}y^3}{xy^{-1}}\right)^5 \right\}$.

6. Find by inspection the characteristics of the logarithms of 3174, 625.7, 3.502, .4, .374, .000135, 23.22065.

7. The mantissa of log 37203 is .5705780: write the logarithms of 37.203, .000037203, 372030000.

8. The logarithm of 7623 is 3.8821259: write the numbers whose logarithms are .8821259, $\bar{6}.8821259$, 7.8821259.

Given $\log 2 = .3010300$, $\log 3 = .4771213$, $\log 7 = .8450980$, find the value of

9. $\log 729$. **10.** $\log 8400$. **11.** $\log .256$.

12. $\log 5.832$. **13.** $\log \sqrt[3]{392}$. **14.** $\log .304\dot{8}$.

15. Show that $\log \frac{11}{15} + \log \frac{490}{297} - 2 \log \frac{7}{9} = \log 2$.

16. Find to six decimal places the value of

$$\log \frac{225}{224} - 2 \log \frac{20}{189} + \log \frac{512}{81}.$$

17. Simplify $\log \{(10.8)^{\frac{1}{2}} \times (.24)^{\frac{5}{3}} \div (90)^{-2}\}$, and find its numerical value.

18. Find the value of

$$\log (\sqrt[3]{126} \cdot \sqrt{108} \div \sqrt[6]{1008} \cdot \sqrt[3]{162}).$$

19. Find the value of $\log \sqrt[5]{\dfrac{588 \times 768}{686 \times 972}}$.

20. Find the number of digits in 42^{42}.

21. Show that $\left(\dfrac{81}{80}\right)^{1000}$ is greater than 100000.

22. How many ciphers are there between the decimal point and the first significant digit in $\left(\dfrac{2}{3}\right)^{1000}$?

23. Find the value of $\sqrt[5]{.01008}$, having given

$$\log 398742 = 5.6006921.$$

24. Find the seventh root of .00792, having given

$$\log 11 = 1.0413927 \text{ and } \log 500.977 = 2.6998179.$$

25. Find the value of $2 \log \frac{75}{49} + \log \frac{135}{32} - 3 \log \frac{45}{28}$.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9008
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

USE OF THE TABLE.

443. On pages 360–361 we give a four-place table containing the mantissæ of the common logarithms of all integers from 100 to 1000.

444. To find the logarithm of a number.

(*a*) Suppose the number consists of three figures, as 56.7. In the column headed N find the first two significant figures. On a line with these and in the column having at the top the third figure will be found the mantissa. Thus on a line with 56 and in the column headed 7 we find 7536. To this, which is the decimal part of the logarithm, prefix the characteristic [Art. 436], and we have

$$\log 56.7 = 1.7536.$$

(*b*) Since in common logarithms the mantissa remains unchanged when the number is multiplied by an integral power of 10, we change one or two-figure numbers into three-figure numbers by addition of ciphers before looking for the mantissæ. The mantissa of log 56 will be that of 560, the only change in the logarithm being in the characteristic.

Thus
$$\log 560 = 2.7482,$$
$$\log \ 56 = 1.7482.$$

In the same manner log 7 has for mantissa that of log 700.

$$\log 700 = 2.8451,$$
$$\log \ \ 7 = 0.8451.$$

(*c*) Suppose the logarithm of a number of more than three figures, as 62543, is required. Since the number lies between 62500 and 62600, its logarithm lies between their logarithms. In the column headed N we find the first two figures, 62; on a line with these and in the columns headed 5, and 6, we find the mantissæ .7959 and .7966. Prefixing the characteristic [Art. 436], we have

$$\log 62600 = 4.7966,$$
$$\log 62500 = 4.7959.$$

Therefore while the number increases from 62500 to 62600, the logarithm increases .0007. Now our number is $\frac{43}{100}$ of the way from 62500 to 62600; hence if to the logarithm of 62500 we add $\frac{43}{100}$ of .0007, a nearly correct logarithm of 62543 is obtained.

Thus

$$\begin{aligned} \log 62543 &= 4.7959 \\ &\ \underline{.0003} \text{ correction} \\ &= 4.7962 \end{aligned}$$

(*d*) Suppose the logarithm of a decimal, as .0005243, is required. The number lies between .0005240 and .0005250. In the column headed N we find the first two significant figures, 52; on a line with these and in the columns headed 4, and 5, we find the mantissæ .7193 and .7202. Prefixing the characteristic [Art. 437], we have

$$\begin{aligned} \log .0005250 &= \bar{4}.7202 \\ \log \underline{.0005240} &= \underline{\bar{4}.7193} \\ \text{differences } .0000010 &\qquad .0009 \end{aligned}$$

Now .0005243 is .0000003 greater than .0005240; hence log .0005243 equals log .0005240 plus $\frac{.0000003}{.0000010}$ or $\frac{3}{10}$ of .0009 (the difference of logarithms);

that is,

$$\begin{aligned} \log .0005243 &= \bar{4}.7193 \\ &\ \underline{.0003} \text{ (nearly)} \\ &= \bar{4}.7196 \end{aligned}$$

In practice *negative characteristics* are usually avoided by adding them to 10 and writing -10 after the logarithm. Thus in the above example $\bar{4}.7196 = 6.7196 - 10$.

445. The increase in the logarithms on the same line, as we pass from column to column, is called the *tabular difference*. In finding the logarithm of 62543, we *assumed* that the differences of logarithms are proportional to the differences of their corresponding numbers, which gives us results that are approximately correct. For greater accuracy we must use tables of more places.

446. To find the number corresponding to a logarithm.

(*a*) Suppose a logarithm, as 1.7466, is given to find the corresponding number.

Look in the table for the mantissa .7466. It is found in the column headed 8 and on the line with 55 in the column headed *N*. Therefore we take the figures 558, and, as the characteristic is 1, point off two places, obtaining the number 55.8.

(*b*) Suppose a logarithm, as 3.7531, is given to find the corresponding number.

The exact mantissa, .7531, is not found in the table, therefore take out the next larger, .7536, and the next smaller, .7528, and retain the characteristic in arranging the work.

$$\begin{array}{lrlr} \text{Thus, the number corresponding to} & 3.7536 & \text{is} & 5670 \\ \text{and the number corresponding to} & \underline{3.7528} & \text{is} & \underline{5660} \\ \text{differences} & .0008 & & 10 \end{array}$$

Now the logarithm 3.7531 is .0003 greater than the logarithm 3.7528, and a difference in logarithms of .0008 corresponds to a difference in numbers of 10; therefore we should increase the number corresponding to the logarithm 3.7528 by $\frac{.0003}{.0008}$ or $\frac{3}{8}$ of 10.

Thus the number corresponding to the logarithm

$$\begin{aligned} 3.7531 &= 5660 \\ &\quad\;\; \underline{3.7} \text{ correction} \\ &= 5663.7 \end{aligned}$$

(*c*) Suppose a logarithm, as 8.8225 − 10 or $\bar{2}.8225$, is given to find the corresponding number.

Take out the mantissæ as in the previous example.

$$\begin{array}{lrlr} \text{The number corresponding to} & \bar{2}.8228 & \text{is} & .0665 \quad \text{[Art. 437.]} \\ \text{The number corresponding to} & \underline{\bar{2}.8222} & \text{is} & \underline{.0664} \\ \text{differences} & .0006 & & .0001 \end{array}$$

Now the logarithm $\bar{2}.8225$ is .0003 greater than the logarithm $\bar{2}.8222$, and a difference in logarithms of .0006 corresponds to a difference in numbers of .0001; therefore we should increase the number corresponding to the logarithm $\bar{2}.8222$ by $\frac{.0003}{.0006}$ or $\frac{3}{6}$ of .0001.

Thus

the number corresponding to the logarithm $\bar{2}.8222 = .0664$
the number corresponding to the logarithm $\bar{2}.8225 = .0664$

Correction, $.00005$

$= .06645$

EXAMPLES XXXVIII. b.

Find the common logarithms of the following:

1. 50. **4.** .341. **7.** 12345.

2. 203. **5.** 0.045. **8.** 0.010203.

3. 6.73. **6.** 5265. **9.** 354.076.

Find the numbers corresponding to the following common logarithms:

10. 1.8156. **12.** 4.0022. **14.** $\bar{3}.8441$.

11. 2.1439. **13.** $\bar{1}.9131$. **15.** $7.4879 - 10$.

447. Cologarithms. The logarithm of the reciprocal of a number is called the *cologarithm* of that number.

Thus $\text{colog } 210 = \log \frac{1}{210} = \log 1 - \log 210$.

Since $\log 1 = 0$, we write it in the form $10 - 10$ and then subtract $\log 210$, which gives

$$\text{colog } 210 = (10 - 2.3222) - 10 = 7.6778 - 10.$$

Hence

RULE. *To find the cologarithm of a number, subtract the logarithm of the number from* 10 *and write* -10 *after the result.*

448. The advantage gained by the use of cologarithms is the substitution of addition for subtraction.

Ex. Find by use of logarithms the value of $\frac{4.26}{7.42 \times .058}$.

$$\begin{aligned}\log\frac{4.26}{7.42 \times .058} &= \log 4.26 + \log\frac{1}{7.42} + \log\frac{1}{.058}\\ &= \log 4.26 + \operatorname{colog} 7.42 + \operatorname{colog} .058\\ &= .6294 + (9.1296 - 10) + 1.2366\\ &= 10.9956 - 10.\end{aligned}$$

The number corresponding to this logarithm is 9.9.

In finding colog .058 we proceed as follows:

$$\begin{aligned}\operatorname{colog} .058 = \log\frac{1}{.058} &= 10 - [\log .058] - 10 \text{ (Art. 447)},\\ &= 10 - [8.7634 - 10] - 10 \text{ (Art. 437)},\\ &= 10 - 8.7634 + 10 - 10 = 1.2366.\end{aligned}$$

449. Exponential Equations. Equations in which the unknown quantity occurs as an exponent are called *exponential equations*, and are readily solved by the aid of logarithms.

Ex. Find the value of x in $15^x = 28$.

Taking the logarithms of both sides of the equation, we have

$$\log 15^x = \log 28;$$

$$\therefore x \log 15 = \log 28.$$

$$x = \frac{\log 28}{\log 15} = \frac{1.4472}{1.1761} = 1.2305 +.$$

EXAMPLES XXXVIII. c.

Find by use of logarithms:

1. $\frac{24.051 \times .02456}{.006705 \times .0203}$.

2.* $\frac{145.206 \times (-7.564)}{448.1 \times (-.2406)(-47.85)}$.

3. $(742.8024)^{\frac{2}{3}}$.

4. $(-.0012045)^{\frac{3}{5}}$.

5. $\frac{\sqrt[3]{4.8} \times \sqrt[4]{.002} \times \sqrt[5]{442.6}}{(18)^2 \times .7^3 \times (3.4562)^{\frac{1}{2}}}$.

6. $\frac{\sqrt{9.8149} \times 80.80008}{\sqrt[7]{8283} \times (.0006412)^4}$.

7. $845692.1 \times .845856$.

8. $.00010101 \times (7117.1)^6$.

9. $\frac{(285.42)^{1.4} \times (5.672)^3}{\sqrt{20} \times \sqrt[3]{.02} \times \sqrt[3]{-124.89}}$.

10. $\sqrt[3]{\frac{12.876 \times \sqrt{.068} \times (.005157)^2}{29.029 \times (52.81)^4 \times (.4)^9}}$.

11. $3^{x+2} = 405$.

12. $10^{5-3x} = 2^{7-2x}$.

13. $12^{3x-4} \times 18^{7-2x} = 1458$.

14. $2^x \times 6^{x-2} = 5^{2x} \times 7^{1-x}$.

15. $2^{x+y} = 6^y$, $3^x = 3 \times 2^{y+1}$.

16. $3^{1-x-y} = 4^{-y}$, $2^{2x-1} = 3^{3y-x}$.

* Treat negative quantities occurring in logarithmic work as positive. When the numerical result is obtained, determine its sign by the ordinary rules of multiplication and division.

CHAPTER XXXIX.

Interest and Annuities.

450. Questions involving Simple Interest are easily solved by the rules of Arithmetic; but in Compound Interest the calculations are often very laborious. We shall now show how these arithmetical calculations may be simplified by the aid of logarithms. Instead of taking as the rate of interest the interest on \$ 100 for one year, it will be found more convenient to take the interest on \$ 1 for one year. If this be denoted by \$ r, and the amount of \$ 1 for 1 year by \$ R, we have $R = 1 + r$.

451. To find the interest and amount of a given sum in a given time at compound interest.

Let P denote the principal, R the amount of \$ 1 in one year, n the number of years, I the interest, and M the amount.

The amount of P at the end of the first year is PR; and, since this is the principal for the second year, the amount at the end of the second year is $PR \times R$ or PR^2. Similarly the amount at the end of the third year is PR^3, and so on; hence the amount in n years is PR^n; that is,

$$M = PR^n;$$

and therefore $$I = P(R^n - 1).$$

Ex. Find the amount of \$ 100 in a hundred years, allowing compound interest at the rate of 5 per cent, payable quarterly; having given

$$\log 2 = .3010300, \quad \log 3 = .4771213, \quad \log 14.3906 = 1.15808.$$

The amount of \$ 1 in a quarter of a year is \$ $(1 + \frac{1}{4} \cdot \frac{5}{100})$ or \$ $\frac{81}{80}$.

The number of payments is 400. If M be the amount, we have

$$M = 100(\tfrac{81}{80})^{400};$$

$$\therefore \log M = \log 100 + 400(\log 81 - \log 80)$$

$$= 2 + 400(4\log 3 - 1 - 3\log 2)$$

$$= 2 + 400(.0053952) = 4.15808;$$

whence $M = 14390.6.$

Thus the amount is \$ 14390.60.

NOTE. At simple interest the amount is \$600.

452. To find the present value and discount of a given sum due in a given time, allowing compound interest.

Let P be the given sum, V the present value, D the discount, R the amount of \$1 for one year, n the number of years.

Since V is the sum which, put out to interest at the present time, will in n years amount to P, we have

$$P = VR^n;$$

$$\therefore V = PR^{-n},$$

and

$$D = P - V = P(1 - R^{-n}).$$

ANNUITIES.

453. An **annuity** is a fixed sum paid periodically under certain stated conditions; the payment may be made either once a year or at more frequent intervals. Unless it is otherwise stated, we shall suppose the payments annual.

454. To find the amount of an annuity left unpaid for a given number of years, allowing compound interest.

Let A be the annuity, R the amount of \$1 for one year, n the number of years, M the amount.

At the end of the first year A is due, and the amount of this sum in the remaining $n-1$ years is AR^{n-1}; at the end of the second year another A is due, and the amount of this sum in the remaining $n-2$ years is AR^{n-2}; and so on.

$$\therefore M = AR^{n-1} + AR^{n-2} + \cdots + AR^2 + AR + A$$

$$= A(1 + R + R^2 + \cdots \text{to } n \text{ terms}) = A\frac{R^n - 1}{R - 1}.$$

455. To find the present value of an annuity to continue for a given number of years, allowing compound interest.

Let A be the annuity, R the amount of $1 in one year, n the number of years, V the required present value.

The present value of A due in 1 year is AR^{-1};
the present value of A due in 2 years is AR^{-2};
the present value of A due in 3 years is AR^{-3}; and so on.
[Art. 452.]

Now V is the sum of the present values of the different payments;

$$\therefore\ V = AR^{-1} + AR^{-2} + AR^{-3} + \cdots \text{ to } n \text{ terms}$$
$$= AR^{-1}\frac{1-R^{-n}}{1-R^{-1}} = A\frac{1-R^{-n}}{R-1}.$$

NOTE. This result may also be obtained by dividing the value of M, given in Art. 454, by R^n. [Art. 451.]

COR. If we make n infinite we obtain for the present value a perpetual annuity

$$V = \frac{A}{R-1} = \frac{A}{r}.$$

EXAMPLES XXXIX.

1. If in the year 1600 a sum of $1000 had been left to accumulate for 300 years, find its amount in the year 1900, reckoning compound interest at 4 per cent per annum. Given

$$\log 104 = 2.0170333 \text{ and } \log 12885.5 = 4.10999.$$

2. Find in how many years a sum of money will amount to one hundred times its value at $5\frac{1}{2}$ per cent per annum compound interest. Given $\log 1055 = 3.023$.

3. Find the present value of $6000 due in 20 years, allowing compound interest at 8 per cent per annum. Given

$$\log 2 = .30103,\ \log 3 = .47712, \text{ and } \log 12875 = 4.10975.$$

4. Find the amount of an annuity of $100 in 15 years, allowing compound interest at 4 per cent per annum. Given

$$\log 1.04 = .01703, \text{ and } \log 180075 = 5.25545.$$

5. What is the present value of an annuity of $1000 due in 30 years, allowing compound interest at 5 per cent per annum?

CHAPTER XL.

Limiting Values and Vanishing Fractions.

456. It will be convenient here to introduce a phraseology and notation which the student will frequently meet with in his mathematical reading.

457. Functions. An expression which involves any quantity, as x, and whose value is dependent on that of x, is called a **function of x.** Functions of x are usually denoted by symbols of the form $f(x)$, $f'(x)$, $F(x)$, $\phi(x)$, and read "the f function of x," "the f' function of x," etc.

Thus the equation $y = f(x)$ may be considered equivalent to a statement that any change made in the value of x will produce a consequent change in y, and *vice versâ*. The quantities x and y are called **variables,** and are further distinguished as the **independent variable** and the **dependent variable.**

An *independent variable* is a quantity which may have any value we choose to assign to it, and the corresponding *dependent variable* has its value determined as soon as the value of the independent variable is known.

458. Definition. If $y = f(x)$, and if when x approaches a value a, the function $f(x)$ can be made to differ by as little as we please from a fixed quantity b, then b is called the **limit** of y when $x = a$.

For instance, if S denote the sum of n terms of the series $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$; then $S = 2 - \frac{1}{2^{n-1}}$.

Here S is a function of n, and $\frac{1}{2^{n-1}}$ can be made as small as we please by increasing n; that is, the limit of S is 2 when n is infinite. This may be expressed by writing $\underset{n=\infty}{\text{limit}}\ S=2$. The sign $\doteq$ is sometimes used instead of the words "*approaches as a limit.*"

459. We shall often have occasion to deal with expressions consisting of a series of terms arranged according to powers of some common letter, such as

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots$$

where the coefficients a_0, a_1, a_2, $a_3 \cdots$ are finite quantities independent of x, and the number of terms may be limited or unlimited.

It will therefore be convenient to discuss some propositions connected with the limiting values of such expressions under certain conditions.

460. Limiting Value. *The limit of the series*

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

when x is indefinitely diminished is a_0.

(i.) Suppose that the series consists of an *infinite* number of terms.

Let b be the greatest of the coefficients a_1, a_2, a_3, $\cdots$; and let us denote the given series by $a_0 + S$; then

$$S < bx + bx^2 + bx^3 + \cdots;$$

and if $x < 1$, we have $\qquad S < \dfrac{bx}{1-x}.$

Thus when x is indefinitely diminished, S can be made as small as we please; hence the limit of the given series is a_0.

(ii.) If the series consists of a *finite* number of terms, S is less than in the case we have considered, hence still more is the proposition true.

461. Value of Any Term. *In the series*

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots,$$

by taking x small enough we may make any term as large as we please compared with the sum of all that follow it; and by taking x large enough we may make any term as large as we please compared with the sum of all that precede it.

(i.) The ratio of any term, as a_nx^n, to the sum of all that follow it is

$$\frac{a_nx^n}{a_{n+1}x^{n+1} + a_{n+2}x^{n+2} + \cdots} \text{ or } \frac{a_n}{a_{n+1}x + a_{n+2}x^2 + \cdots}$$

When x is indefinitely small, the denominator can be made as small as we please; that is, the fraction can be made as large as we please.

(ii.) Again, the ratio of the term a_nx^n to the sum of all that precede it is

$$\frac{a_nx^n}{a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots}, \text{ or } \frac{a_n}{a_{n-1}y + a_{n-2}y^2 + \cdots};$$

where $y = \dfrac{1}{x}$.

When x is indefinitely large, y is indefinitely small; hence, as in the previous case, the fraction can be made as large as we please.

462. The following particular form of the foregoing proposition is very useful.

In the expression

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

consisting of a finite number of terms in *descending* powers of x, by taking x small enough the last term a_0 can be made as large as we please compared with the sum of all the terms that precede it, and by taking x large enough the first term a_nx^n can be made as large as we please compared with the sum of all that follow it.

Ex. 1. By taking n large enough we can make the first term of $n^4 - 5n^3 - 7n + 9$ as large as we please compared with the sum of all the other terms; that is, we may take the first term n^4 as the equivalent of the whole expression, with an error as small as we please provided n be taken large enough.

Ex. 2. Find the limit of $\dfrac{3x^3 - 2x^2 - 4}{5x^3 - 4x + 8}$ when (1) x is infinite; (2) x is zero.

(1) In the numerator and denominator *we may disregard all terms but the first;* hence

$$\underset{x=\infty}{\text{limit}} \frac{3x^3 - 2x^2 - 4}{5x^3 - 4x + 8} = \frac{3x^3}{5x^3} = \frac{3}{5}.$$

(2) When x is indefinitely small *we may disregard all terms but the last;* hence the limit is $\dfrac{-4}{8}$, or $-\dfrac{1}{2}$.

VANISHING FRACTIONS.

463. Suppose it is required to find the limit of

$$\frac{x^2 + ax - 2a^2}{x^2 - a^2}$$

when $x = a$.

If we put $x = a + h$, then h will approach the value zero as x approaches the value a.

Substituting $a + h$ for x,

$$\frac{x^2 + ax - 2a^2}{x^2 - a^2} = \frac{3ah + h^2}{2ah + h^2} = \frac{3a + h}{2a + h};$$

and when h is indefinitely small the limit of this expression is $\frac{3}{2}$.

There is, however, another way of regarding the question; for

$$\frac{x^2 + ax - 2a^2}{x^2 - a^2} = \frac{(x-a)(x+2a)}{(x-a)(x+a)} = \frac{x+2a}{x+a},$$

and if we *now* put $x = a$ the value of the expression is $\frac{3}{2}$, as before.

If in the given expression $\dfrac{x^2 + ax - 2a^2}{x^2 - a^2}$ we put $x = a$ *before* simplification, it will be found that it assumes the form $\frac{0}{0}$, the value of which is indeterminate [Art. 183]; also we see that it has this form in consequence of the factor

$x-a$ appearing in both numerator and denominator. Now we cannot divide by a *zero factor*, but as long as x is not absolutely equal to a, the factor $x-a$ may be removed, and we then find that the nearer x approaches to the value a, the nearer does the value of the fraction approximate to $\frac{3}{2}$, or in accordance with the definition of Art. 458,

$$\text{when } x=a, \text{ the limit of } \frac{x^2+ax-2a^2}{x^2-a^2} \text{ is } \frac{3}{2}.$$

464. Vanishing Fractions. If $f(x)$ and $f'(x)$ are two functions of x, each of which becomes equal to zero for some particular value a of x, the fraction $\dfrac{f(a)}{f'(a)}$ takes the form $\frac{0}{0}$, and is called a *vanishing fraction.*

Ex. 1. If $x=3$, find the limit of

$$\frac{x^3-5x^2+7x-3}{x^3-x^2-5x-3}.$$

When $x=3$, the expression reduces to the indeterminate form $\frac{0}{0}$; but by removing the factor $x-3$ from numerator and denominator, the fraction becomes $\dfrac{x^2-2x+1}{x^2+2x+1}$. When $x=3$, this reduces to $\frac{1}{4}$, which is therefore the required limit.

Ex. 2. The fraction $\dfrac{\sqrt{3x-a}-\sqrt{x+a}}{x-a}$ becomes $\dfrac{0}{0}$ when $x=a$.

To find its limit, multiply numerator and denominator by the surd conjugate to $\sqrt{3x-a}-\sqrt{x+a}$; the fraction then becomes

$$\frac{(3x-a)-(x+a)}{(x-a)(\sqrt{3x-a}+\sqrt{x+a})}, \text{ or } \frac{2}{\sqrt{3x-a}+\sqrt{x+a}};$$

whence by putting $x=a$ we find that the limit is $\dfrac{1}{\sqrt{2a}}$.

Ex. 3. The fraction $\dfrac{1-\sqrt[3]{x}}{1-\sqrt[5]{x}}$ becomes $\dfrac{0}{0}$ when $x=1$.

To find its limit, put $x=1+h$, and expand by the Binomial Theorem. Thus the fraction

$$=\frac{1-(1+h)^{\frac{1}{3}}}{1-(1+h)^{\frac{1}{5}}}=\frac{1-(1+\frac{1}{3}h-\frac{1}{9}h^2+\cdots)}{1-(1+\frac{1}{5}h-\frac{2}{25}h^2+\cdots)}=\frac{-\frac{1}{3}+\frac{1}{9}h-\cdots}{-\frac{1}{5}+\frac{2}{25}h-\cdots}.$$

Now $h=0$ when $x=1$; hence the required limit is $\frac{5}{3}$.

465. We shall now discuss some peculiarities which may arise in the solution of a quadratic equation.

Let the equation be

$$ax^2 + bx + c = 0.$$

If $c = 0$, then $\quad ax^2 + bx = 0;$

whence $\quad x = 0$, or $-\frac{b}{a};$

that is, one of the roots is zero and the other is finite.

If $b = 0$, the roots are equal in magnitude and opposite in sign.

If $a = 0$, the equation reduces to $bx + c = 0$; and it appears that in this case the quadratic furnishes only one root, namely, $-\frac{c}{b}$. But every quadratic equation has two roots, and in order to discuss the value of the other root we proceed as follows:

Write $\frac{1}{y}$ for x in the original equation and clear of fractions; thus, $\quad cy^2 + by + a = 0.$

Now put $a = 0$, and we have

$$cy^2 + by = 0;$$

the solution of which is $y = 0$, or $-\frac{b}{c}$; that is, $x = \infty$, or $-\frac{c}{b}$.

Hence, *in any quadratic equation one root will become infinite if the coefficient of x^2 becomes zero.*

This is the form in which the result will be most frequently met with in other branches of higher Mathematics, but the student should notice that it is merely a convenient abbreviation of the following fuller statement:

In the equation $ax^2 + bx + c = 0$, if a is very small, one root is very large, and as a is indefinitely diminished this root becomes indefinitely great. In this case the finite root approximates to $-\frac{c}{b}$ as its limit.

EXAMPLES XL.

Find the limits of the following expressions:

(1) when $x = \infty$. (2) when $x = 0$.

1. $\dfrac{(2x-3)(3-5x)}{7x^2-6x+4}$.

2. $\dfrac{(3x^2-1)^2}{x^4+9}$.

3. $\dfrac{(3+2x^3)(x-5)}{(4x^3-9)(1+x)}$.

4. $\dfrac{(x-3)(2-5x)(3x+1)}{(2x-1)^3}$.

5. $\dfrac{1-x^2}{2x^3-1} \div \dfrac{1-x}{2x^2}$.

6. $\dfrac{(3-x)(x+5)(2-7x)}{(7x-1)(x+1)^3}$.

Find the limits of

7. $\dfrac{x^3+1}{x^2-1}$, when $x = -1$.

8. $\dfrac{\sqrt{x}-\sqrt{2a}+\sqrt{x-2a}}{\sqrt{x^2-4a^2}}$, when $x = 2a$.

9. $\dfrac{(a^2-x^2)^{\frac{1}{2}}+(a-x)^{\frac{3}{2}}}{(a^3-x^3)^{\frac{1}{2}}+(a-x)^{\frac{1}{2}}}$, when $x = a$.

10. $\dfrac{\sqrt{a^2+ax+x^2}-\sqrt{a^2-ax+x^2}}{\sqrt{a+x}-\sqrt{a-x}}$, when $x = 0$.

CHAPTER XLI.

CONVERGENCY AND DIVERGENCY OF SERIES.

466. We have, in Chapter XXXIV., defined a **series** as an expression in which the successive terms are formed by some regular law; if the series terminates at some assigned term, it is called a **finite series**; if the number of terms is unlimited, it is called an **infinite series.**

In the present chapter, we shall usually denote a series by an expression of the form

$$u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

467. DEFINITIONS. Suppose that we have a series consisting of n terms. The sum of the series will be a function of n; if n increases indefinitely, the sum either tends to become equal to a certain finite *limit,* or else it becomes infinitely great.

An infinite series is said to be **convergent** when the sum of the first n terms cannot numerically exceed some finite quantity, however great n may be.

An infinite series is said to be **divergent** when the sum of the first n terms can be made numerically greater than any finite quantity by taking n sufficiently great.

TESTS FOR CONVERGENCY.

468. When the Sum of the First n Terms of a Given Series is Known. If we can find the sum of the first n terms of a given series, we may ascertain whether it is convergent or divergent by examining whether the series remains finite, or becomes infinite, when n is made indefinitely great

For example, the sum of the first n terms of the series

$$1 + x + x^2 + x^3 + \cdots \text{ is } \frac{1 - x^n}{1 - x}.$$

If x is numerically less than 1, the sum approaches to the finite limit $\dfrac{1}{1-x}$, and the series is therefore *convergent.*

If x is numerically greater than 1, the sum of the first n terms is $\dfrac{x^n - 1}{x - 1}$, and by taking n sufficiently great, this can be made greater than any finite quantity; thus the series is *divergent.*

If $x = 1$, the sum of the first n terms is n, and therefore the series is *divergent.*

If $x = -1$, the series becomes

$$1 - 1 + 1 - 1 + 1 - 1 + \cdots.$$

The sum of an even number of terms is zero, while the sum of an odd number of terms is 1; and thus the sum oscillates between the values 0 and 1. This series belongs to a class which may be called *oscillating* or *periodic convergent series.*

469. When the Sum of the First *n* Terms of a Given Series is Unknown. There are many cases in which we have no method of finding the sum of the first n terms of a series. We proceed therefore to investigate rules by which we can test the convergency or divergency of a given series without effecting its summation.

470. First Test. *An infinite series in which the terms are alternately positive and negative is convergent if each term is numerically less than the preceding term.*

Let the series be denoted by

$$u_1 - u_2 + u_3 - u_4 + u_5 - u_6 + \cdots$$

where $$u_1 > u_2 > u_3 > u_4 > u_5 \cdots.$$

The given series may be written in each of the following forms:

$$(u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \quad . \quad . \quad . \quad . \quad (1),$$

$$u_1 - (u_2 - u_3) - (u_4 - u_5) - (u_6 - u_7) - \quad . \quad . \quad . \quad (2).$$

From (1) we see that the sum of any number of terms is a positive quantity; and from (2) that the sum of any number of terms is less than u_1; hence the series is convergent.

For example, in the series

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots$$

the terms are alternately positive and negative, and each term is numerically less than the preceding one; hence the series is convergent.

471. Second Test. *An infinite series in which all the terms are of the same sign is divergent if each term is greater than some finite quantity, however small.*

For if each term is greater than some finite quantity a, the sum of the first n terms is greater than na; and this, by taking n sufficiently great, can be made to exceed any finite quantity.

472. Before proceeding to investigate further tests of convergency and divergency, we shall lay down two important principles, which may almost be regarded as axioms.

I. If a series is convergent it will remain convergent, and if divergent it will remain divergent, when we add or remove any *finite* number of its terms; for the sum of these terms is a finite quantity.

II. If a series in which all the terms are positive is convergent, then the series is convergent when some or all of the terms are negative; for the sum is clearly greatest when all the terms have the same sign.

We shall suppose that all the terms are positive unless the contrary is stated.

473. Third Test. *An infinite series is convergent if from and after some fixed term the ratio of each term to the preceding term is numerically less than some quantity which is itself numerically less than unity.*

Let the series beginning from the fixed term be denoted by

$$u_1 + u_2 + u_3 + u_4 + \cdots;$$

and let $$\frac{u_2}{u_1} < r, \ \frac{u_3}{u_2} < r, \ \frac{u_4}{u_3} < r, \ \cdots,$$

where $r < 1$.

Then $$u_1 + u_2 + u_3 + u_4 + \cdots$$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \frac{u_4}{u_3} \cdot \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \cdots\right)$$

$$< u_1(1 + r + r^2 + r^3 + \cdots);$$

that is, $< \frac{u_1}{1-r}$, since $r < 1$.

Hence the given series is convergent.

474. In the enunciation of the preceding article the student should notice the significance of the words "from and after a fixed term."

Consider the series

$$1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots.$$

Here $$\frac{u_n}{u_{n-1}} = \frac{nx}{n-1} = \left(1 + \frac{1}{n-1}\right)x;$$

and by taking n sufficiently large we can make this ratio approximate to x as nearly as we please, and the ratio of each term to the preceding term will ultimately be x. Hence if $x < 1$, the series is convergent.

But the ratio $\frac{u_n}{u_{n-1}}$ will not be less than 1, until $\frac{nx}{n-1} < 1$;

that is, until $n > \frac{1}{1-x}$.

Here we have a case of a convergent series in which the terms may increase up to a certain point, and then begin to decrease. For example, if $x = \frac{99}{100}$, then $\frac{1}{1-x} = 100$, and the terms do not begin to decrease until after the 100th term.

475. Fourth Test. *An infinite series in which all the terms are of the same sign is divergent if from and after some fixed term the ratio of each term to the preceding term is greater than unity, or equal to unity.*

Let the fixed term be denoted by u_1. If the ratio is equal to unity, each of the succeeding terms is equal to u_1, and the sum of n terms is equal to nu_1; hence the series is divergent.

If the ratio is greater than unity, each of the terms after the fixed term is greater than u_1, and the sum of n terms is greater than nu_1; hence the series is divergent.

476. In the practical application of these tests, to avoid having to ascertain the particular term after which each term is greater or less than the preceding term, it is convenient to find the limit of $\frac{u_n}{u_{n-1}}$ when n is indefinitely increased; let this limit be denoted by l.

If $l < 1$, the series is convergent. [Art. 473.]

If $l > 1$, the series is divergent. [Art. 475.]

If $l = 1$, the series may be either convergent or divergent, and a further test will be required; for it may happen that $\frac{u_n}{u_{n-1}} < 1$, *but continually approaching to* 1 *as its limit when* n *is indefinitely increased.* In this case we cannot name any finite quantity r which is itself less than 1 and yet greater than l. Hence the test of Art. 473 fails. If, however, $\frac{u_n}{u_{n-1}} > 1$, but continually approaching to 1 as its limit, the series is divergent by Art. 475.

We shall use "Lim $\frac{u_n}{u_{n-1}}$" as an abbreviation of the words "the limit of $\frac{u_n}{u_{n-1}}$ when n is infinite."

Ex. 1. Find whether the series whose nth term is $\dfrac{(n+1)x^n}{n^2}$ is convergent or divergent.

Here $$\frac{u_n}{u_{n-1}} = \frac{(n+1)x^n}{n^2} \div \frac{nx^{n-1}}{(n-1)^2} = \frac{(n+1)(n-1)^2}{n^3} \cdot x;$$

$$\therefore \text{Lim} \frac{u_n}{u_{n-1}} = x;$$

hence if $x < 1$ the series is convergent;
if $x > 1$ the series is divergent.

If $x = 1$, then $\text{Lim} \dfrac{u_n}{u_{n-1}} = 1$, and a further test is required.

Ex. 2. Is the series
$$1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots$$
convergent or divergent?

Here $$\text{Lim} \frac{u_n}{u_{n-1}} = \text{Lim} \frac{n^2x^{n-1}}{(n-1)^2x^{n-2}} = x.$$

Hence if $x < 1$ the series is convergent;
if $x > 1$ the series is divergent.

If $x = 1$ the series becomes $1^2 + 2^2 + 3^2 + 4^2 + \cdots$, and is obviously divergent.

Ex. 3. In the series
$$a + (a+d)r + (a+2d)r^2 + \cdots + (a + \overline{n-1} \cdot d)r^{n-1} + \cdots,$$
$$\text{Lim} \frac{u_n}{u_{n-1}} = \text{Lim} \frac{a + (n-1)d}{a + (n-2)d} \cdot r = r;$$
thus if $r < 1$ the series is convergent, and the sum is finite.

477. Fifth Test. *If there are two infinite series in each of which all the terms are positive, and if the ratio of the corresponding terms in the two series is always finite, the two series are both convergent, or both divergent.*

Let the two infinite series be denoted by
$$u_1 + u_2 + u_3 + u_4 + \cdots,$$
and $$v_1 + v_2 + v_3 + v_4 + \cdots.$$

The value of the fraction
$$\frac{u_1 + u_2 + u_3 + \cdots + u_n}{v_1 + v_2 + v_3 + \cdots + v_n}$$

lies between the greatest and least of the fractions

$$\frac{u_1}{v_1}, \quad \frac{u_2}{v_2}, \quad \cdots \frac{u_n}{v_n},$$

and is therefore a *finite* quantity, L say;

$$\therefore\ u_1 + u_2 + u_3 + \cdots + u_n = L(v_1 + v_2 + v_3 + \cdots + v_n).$$

Hence if one series is finite in value, so is the other; if one series is infinite in value, so is the other; which proves the proposition.

478. Auxiliary Series. The application of the principle of the preceding article is very important, for by means of it we can compare a given series with an *auxiliary series* whose convergency or divergency has been already established. The series discussed in the next article will frequently be found useful as an auxiliary series.

479. The infinite series $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$ is always divergent except when p is positive and greater than 1.

Case I. Let $p > 1$.

The first term is 1; the next two terms together are less than $\frac{2}{2^p}$; the following four terms together are less than $\frac{4}{4^p}$; the following eight terms together are less than $\frac{8}{8^p}$; and so on. Hence the series is less than

$$1 + \frac{2}{2^p} + \frac{4}{4^p} + \frac{8}{8^p} + \cdots;$$

that is, less than a geometrical progression whose common ratio $\frac{2}{2^p}$ is less than 1, since $p > 1$; hence the series is convergent.

Case II. Let $p = 1$.

The series now becomes $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$.

The third and fourth terms together are greater than $\frac{2}{4}$ or $\frac{1}{2}$; the following four terms together are greater than $\frac{4}{8}$ or

$\frac{1}{2}$; the following eight terms together are greater than $\frac{8}{16}$ or $\frac{1}{2}$; and so on. Hence the series is greater than

$$1 + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \tfrac{1}{2} + \cdots,$$

and is therefore divergent. [Art. 475.]

CASE III. Let $p < 1$, or negative.

Each term is now greater than the corresponding term in Case II., therefore the series is divergent.

Hence the series is always divergent except in the case when p is positive and greater than unity.

Ex. Prove that the series $\frac{2}{1} + \frac{3}{4} + \frac{4}{9} + \cdots + \frac{n+1}{n^2} + \cdots$ is divergent.

Compare the given series with $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$.

Thus if u_n and v_n denote the nth terms of the given series and the auxiliary series respectively, we have

$$\frac{u_n}{v_n} = \frac{n+1}{n^2} \div \frac{1}{n} = \frac{n+1}{n};$$

hence $\text{Lim}\frac{u_n}{v_n} = 1$, and therefore the two series are both convergent or both divergent. But the auxiliary series is divergent, therefore also the given series is divergent.

This completes the solution of Ex. 1. [Art. 476.]

480. Convergency of the Binomial Series. *To show that the expansion of* $(1 + x)^n$ *by the Binomial Theorem is convergent when* $x < 1$.

Let u_r, u_{r+1} represent the rth and $(r + 1)$th terms of the expansion; then

$$\frac{u_{r+1}}{u_r} = \frac{n - r + 1}{r}x.$$

When $r > n + 1$, this ratio is negative; that is, from this point the terms are alternately positive and negative when x is positive, and always of the same sign when x is negative.

Now when r is infinite, $\text{Lim}\frac{u_{r+1}}{u_r} = x$ numerically; therefore since $x < 1$ the series is convergent if all the terms are of the same sign; and therefore still more is it convergent when some of the terms are positive and some negative. [Art. 472.]

EXAMPLES XLI.

Find whether the following series are convergent or divergent:

1. $\dfrac{1}{x} - \dfrac{1}{x+a} + \dfrac{1}{x+2a} - \dfrac{1}{x+3a} + \cdots,$

x and a being positive quantities.

2. $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \dfrac{1}{4\cdot 5} + \cdots.$

3. $\dfrac{1}{xy} - \dfrac{1}{(x+1)(y+1)} + \dfrac{1}{(x+2)(y+2)} - \dfrac{1}{(x+3)(y+3)} + \cdots,$

x and y being positive quantities.

4. $\dfrac{x}{1\cdot 2} + \dfrac{x^2}{2\cdot 3} + \dfrac{x^3}{3\cdot 4} + \dfrac{x^4}{4\cdot 5} + \cdots.$

5. $\dfrac{x}{1\cdot 2} + \dfrac{x^2}{3\cdot 4} + \dfrac{x^3}{5\cdot 6} + \dfrac{x^4}{7\cdot 8} + \cdots.$

6. $1 + \dfrac{2^2}{\underline{|2}} + \dfrac{3^2}{\underline{|3}} + \dfrac{4^2}{\underline{|4}} + \cdots.$

7. $\sqrt{\tfrac{1}{2}} + \sqrt{\tfrac{2}{3}} + \sqrt{\tfrac{3}{4}} + \sqrt{\tfrac{4}{5}} + \cdots.$

8. $1 + 3x + 5x^2 + 7x^3 + 9x^4 + \cdots.$

9. $\dfrac{2}{1^p} + \dfrac{3}{2^p} + \dfrac{4}{3^p} + \dfrac{5}{4^p} + \cdots.$

10. $1 + \dfrac{x}{2} + \dfrac{x^2}{5} + \dfrac{x^3}{10} + \cdots + \dfrac{x^n}{n^2+1} + \cdots.$

NOTE. For further information on the subject of Convergency and Divergency of Series the reader may consult Hall & Knight's Higher Algebra, Chapter XXI.

CHAPTER XLII.

UNDETERMINED COEFFICIENTS.

FUNCTIONS OF FINITE DIMENSIONS.

481. In Art. 105 it was proved that if any rational integral function of x vanishes when $x = a$, it is divisible by $x - a$.

Let $$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n$$
be a rational integral function of x of n dimensions, which vanishes when x is equal to each of the unequal quantities

$$a_1,\ a_2,\ a_3,\ \cdots\ a_n.$$

Denote the function by $f(x)$; then since $f(x)$ is divisible by $x - a_1$, we have

$$f(x) = (x - a_1)(p_0x^{n-1} + \cdots),$$

the quotient being of $n - 1$ dimensions.

Similarly, since $f(x)$ is divisible by $x - a_2$, we have

$$p_0x^{n-1} + \cdots = (x - a_2)(p_0x^{n-2} + \cdots),$$

the quotient being of $n - 2$ dimensions; and

$$p_0x^{n-2} + \cdots = (x - a_3)(p_0x^{n-3} + \cdots).$$

Proceeding in this way, we shall finally obtain after n divisions

$$f(x) = p_0(x - a_1)(x - a_2)(x - a_3)\cdots(x - a_n).$$

482. If a rational integral function of *n* dimensions vanishes for more than *n* values of the variable, the coefficient of each power of the variable must be zero.

Let the function be denoted by $f(x)$, where

$$f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n;$$

and suppose that $f(x)$ vanishes when x is equal to each of the unequal values $a_1, a_2, a_3, \cdots a_n$; then

$$f(x) = p_0(x - a_1)(x - a_2)(x - a_3)\cdots(x - a_n).$$

Let c be another value of x which makes $f(x)$ vanish; then since $f(c) = 0$, we have

$$p_0(c - a_1)(c - a_2)(c - a_3)\cdots(c - a_n) = 0;$$

and therefore $p_0 = 0$, since, by hypothesis, none of the other factors is equal to zero. Hence $f(x)$ reduces to

$$p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \cdots + p_n.$$

By hypothesis this expression vanishes for more than n values of x, and therefore $p_1 = 0$.

In a similar manner we may show that each of the coefficients $p_2, p_3, \cdots p_n$ must be equal to zero.

This result may also be enunciated as follows:

If a rational integral function of n dimensions vanishes for more than n values of the variable, it must vanish for every value of the variable.

Cor. If the *function* $f(x)$ vanishes for more than n values of x, the *equation* $f(x) = 0$ has more than n roots.

Hence also, *if an equation of n dimensions has more than n roots it is an identity.*

Ex. Prove that

$$\frac{(x-b)(x-c)}{(a-b)(a-c)} + \frac{(x-c)(x-a)}{(b-c)(b-a)} + \frac{(x-a)(x-b)}{(c-a)(c-b)} = 1.$$

This equation is of *two* dimensions, and it is evidently satisfied by each of the *three* values a, b, c; hence it is an identity.

483. **If two rational integral functions of n dimensions are equal for more than n values of the variable, they are equal for every value of the variable.**

Suppose that the two functions

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n,$$

$$q_0x^n + q_1x^{n-1} + q_2x^{n-2} + \cdots + q_n,$$

are equal for more than n values of x; then the expression

$$(p_0 - q_0)x^n + (p_1 - q_1)x^{n-1} + (p_2 - q_2)x^{n-2} + \cdots + (p_n - q_n)$$

vanishes for more than n values of x; and therefore, by the preceding article,

$$p_0 - q_0 = 0,\ p_1 - q_1 = 0,\ p_2 - q_2 = 0, \cdots p_n - q_n = 0;$$

that is, $$p_0 = q_0,\ p_1 = q_1,\ p_2 = q_2, \cdots p_n = q_n.$$

Hence the two expressions are *identical*, and therefore are equal for every value of the variable. Thus

If two rational integral functions are identically equal, **we may equate the coefficients of the like powers of the variable.**

Cor. This proposition still holds if one of the functions is of lower dimensions than the other. For instance, if

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \cdots + p_n$$
$$= q_2x^{n-2} + q_3x^{n-3} + \cdots + q_n,$$

we have only to suppose that in the above investigation $q_0 = 0$, $q_1 = 0$, and then we obtain

$$p_0 = 0,\ p_1 = 0,\ p_2 = q_2,\ p_3 = q_3, \cdots p_n = q_n.$$

484. The theorem established in the preceding article for functions of finite dimensions is usually referred to as the **Principle of Undetermined Coefficients.** The application of this principle is illustrated in the following examples.

Ex. 1. Find the sum of the series

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1).$$

Assume that

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = A + Bn + Cn^2 + Dn^3 + En^4 + \cdots, \quad (1)$$

where A, B, C, D, E, $\cdots$ are quantities independent of n, whose values have to be determined. Change n into $n+1$; then

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) + (n+1)(n+2)$$
$$= A + B(n+1) + C(n+1)^2 + D(n+1)^3 + E(n+1)^4 + \cdots. \quad (2)$$

By subtracting (1) from (2),

$$(n+1)(n+2) = B + C(2n+1) + D(3n^2 + 3n + 1)$$
$$+ E(4n^3 + 6n^2 + 4n + 1) + \cdots.$$

This equation being true for all integral values of n, the coefficients of the respective powers of n on each side must be equal; thus E and all succeeding coefficients must be equal to zero, and

$$3D = 1;\ 3D + 2C = 3;\ D + C + B = 2;$$

whence $$D = \tfrac{1}{3},\ C = 1,\ B = \tfrac{2}{3}.$$

Hence the sum $\qquad = A + \dfrac{2n}{3} + n^2 + \dfrac{1}{3}n^3.$

To find A, put $n = 1$; the series then reduces to its first term, and

$$2 = A + 2, \text{ or } A = 0.$$

Hence $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{1}{3}n(n+1)(n+2)$.

NOTE. It will be seen from this example that when the nth term is a rational integral function of n, it is sufficient to assume for the sum a function of n which is of one dimension higher than the nth term of the series.

Ex. 2. Find the conditions that $x^3 + px^2 + qx + r$ may be divisible by

$$x^2 + ax + b.$$

The quotient will contain two terms; namely, x and a term independent of x. Hence, we assume

$$x^3 + px^2 + qx + r = (x + k)(x^2 + ax + b).$$

Equating the coefficients of the like powers of x, we have

$$k + a = p, \quad ak + b = q, \quad kb = r.$$

From the last equation $k = \dfrac{r}{b}$; hence by substitution we obtain

$$\frac{r}{b} + a = p, \text{ and } \frac{ar}{b} + b = q;$$

that is, $\qquad r = b(p - a), \text{ and } ar = b(q - b);$

which are the conditions required.

EXAMPLES XLII. a.

Find by the method of Undetermined Coefficients the sum of

1. $1^2 + 3^2 + 5^2 + 7^2 + \cdots$ to n terms.

2. $1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + \cdots$ to n terms.

3. $1 \cdot 2^2 + 2 \cdot 3^2 + 3 \cdot 4^2 + 4 \cdot 5^2 + \cdots$ to n terms.

4. $1^3 + 3^3 + 5^3 + 7^3 + \cdots$ to n terms.

5. $1^4 + 2^4 + 3^4 + 4^4 + \cdots$ to n terms.

6. Find the condition that $x^3 - 3px + 2q$ may be divisible by a factor of the form $x^2 + 2ax + a^2$.

7. Find the conditions that $ax^3 + bx^2 + cx + d$ may be a perfect cube.

8. Find the conditions that $a^2x^4 + bx^3 + cx^2 + dx + f^2$ may be a perfect square.

9. Prove the identity

$$\frac{a^2(x-b)(x-c)}{(a-b)(a-c)} + \frac{b^2(x-c)(x-a)}{(b-c)(b-a)} + \frac{c^2(x-a)(x-b)}{(c-a)(c-b)} = x^2.$$

FUNCTIONS OF INFINITE DIMENSIONS.

485. If the infinite series $a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$ is equal to zero for every finite value of x for which the series is convergent, then each coefficient must be equal to zero identically.

Let the series be denoted by S, and let S_1 stand for the expression $a_1 + a_2x + a_3x^2 + \cdots$; then $S = a_0 + xS_1$, and therefore, by hypothesis, $a_0 + xS_1 = 0$ for all finite values of x. But since S is convergent, S_1 cannot exceed some finite limit; therefore by taking x small enough, xS_1 may be made as small as we please. In this case the limit of S is a_0; but S is *always* zero, therefore a_0 must be equal to zero identically.

Removing the term a_0, we have $xS_1 = 0$ for all finite values of x; that is, $a_1 + a_2x + a_3x^2 + \cdots$ vanishes for all finite values of x.

Similarly, we may prove in succession that each of the coefficients $a_1, a_2, a_3, \cdots$ is equal to zero identically.

486. If two infinite series are convergent and equal to one another for every finite value of the variable, the coefficients of like powers of the variable in the two series are equal.

Suppose that the two series are denoted by

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

and

$$A_0 + A_1x + A_2x^2 + A_3x^3 + \cdots;$$

then the expression

$$a_0 - A_0 + (a_1 - A_1)x + (a_2 - A_2)x^2 + (a_3 - A_3)x^3 + \cdots$$

vanishes for all values of x within the assigned limits; therefore by the last article

$$a_0 - A_0 = 0,\ a_1 - A_1 = 0,\ a_2 - A_2 = 0,\ a_3 - A_3 = 0,\ \cdots,$$

that is, $\quad a_0 = A_0,\ a_1 = A_1,\ a_2 = A_2,\ a_3 = A_3,\ \cdots;$

hence the *coefficients of like powers of the variable are equal,* which proves the proposition.

EXPANSION OF FRACTIONS INTO SERIES.

487. Expand $\dfrac{2+x^2}{1+x-x^2}$ in a series of ascending powers of x.

Let $$\frac{2+x^2}{1+x-x^2} = A + Bx + Cx^2 + Dx^3 + \cdots,$$

where $A, B, C, D, \cdots,$ are constants whose values are to be determined; then

$$\begin{aligned} 2 + x^2 &= A(1+x-x^2) + Bx(1+x-x^2) + Cx^2(1+x-x^2) \\ &\qquad + Dx^3(1+x-x^2) + \cdots \\ &= \begin{array}{r|r|r|l} A + B & x + \quad C & x^2 + \quad D & x^3 + \cdots \\ A & B & C & \\ & -A & -B & \end{array} \end{aligned}$$

Equating the coefficients of like powers of x, we have

$$A = 2, \quad B + A = 0, \quad C + B - A = 1, \quad D + C - B = 0,$$
$$\therefore B = -2; \qquad \therefore C = 5; \qquad \therefore D = -7;$$

thus $$\frac{2+x^2}{1+x-x^2} = 2 - 2x + 5x^2 - 7x^3 + \cdots.$$

488. Both numerator and denominator should be arranged with reference to the ascending powers of the same quantity; then dividing the first term of the numerator by the first term of the denominator determines the *form of the expansion.*

Ex. Expand $\dfrac{2}{2x^2-3x^3}$ in a series of ascending powers of x.

Dividing x^0, of the first term of the numerator, by x^2, of the first term of the denominator, we obtain x^{-2} for the first term of the expansion; therefore we assume

$$\frac{2}{2x^2-3x^3}=Ax^{-2}+Bx^{-1}+C+Dx+\cdots;$$

then
$$2=2A+\left.\begin{matrix}2B\\-3A\end{matrix}\right|x+\left.\begin{matrix}2C\\-3B\end{matrix}\right|x^2+\left.\begin{matrix}2D\\-3C\end{matrix}\right|x^3+\cdots$$

Equating the coefficients of like powers of x, we have

$$A=1;\quad 2B-3A=0,\quad 2C-3B=0,\quad 2D-3C=0,$$
$$\therefore B=\tfrac{3}{2};\qquad \therefore C=\tfrac{9}{4};\qquad \therefore D=\tfrac{27}{8};$$

thus
$$\frac{2}{2x^2-3x^3}=x^{-2}+\tfrac{3}{2}x^{-1}+\tfrac{9}{4}+\tfrac{27}{8}x+\cdots$$
$$=\frac{1}{x^2}+\frac{3}{2x}+\frac{9}{4}+\frac{27x}{8}+\cdots$$

EXAMPLES XLII. b.

Expand to four terms in ascending powers of x:

1. $\dfrac{1+2x}{1-x-x^2}$.
2. $\dfrac{1-8x}{1-x-6x^2}$.
3. $\dfrac{1+x}{2+x+x^2}$.
4. $\dfrac{3+x}{2-x-x^2}$.
5. $\dfrac{1}{1+ax-ax^2-x^3}$.
6. $\dfrac{2-2x+3x^2}{4+x+x^2}$.
7. $\dfrac{1}{2x-3x^2}$.
8. $\dfrac{c}{b-ax}$.
9. $\dfrac{2+x}{3x^2+x^3}$.
10. $\dfrac{1+x+x^2}{x+x^3+x^4}$.

EXPANSION OF SURDS INTO SERIES.

489. Expand $\sqrt{1+x}$ in ascending powers of x.

Let $\sqrt{1+x}=A+Bx+Cx^2+Dx^3+Ex^4+\cdots$.

By squaring both sides of the equation, we have

$$1+x=A^2+2AB\,\Big|\,x+\left.\begin{matrix}B^2\\+2AC\end{matrix}\right|x^2+\left.\begin{matrix}2AD\\+2BC\end{matrix}\right|x^3+\left.\begin{matrix}C^2\\+2BD\\+2AE\end{matrix}\right|x^4+\cdots$$

Equating the coefficients of like powers of x, we have

$$A=1;\quad 2AB=1,\quad B^2+2AC=0,\quad 2AD+2BC=0,$$
$$\therefore B=\tfrac{1}{2};\qquad \therefore C=-\tfrac{1}{8};\qquad \therefore D=\tfrac{1}{16};$$
$$C^2+2BD+2AE=0;$$
$$\therefore E=-\tfrac{5}{128};$$

thus $$\sqrt{1+x}=1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}-\frac{5x^4}{128}+\cdots.$$

NOTE. The expansion can be readily effected by the use of the Binomial Theorem [Art. 421].

EXAMPLES XLII. c.

Expand the following expressions to four terms:

1. $\sqrt{1-x}$.
2. $\sqrt{a-x}$.
3. $\sqrt{a^2-x^2}$.
4. $\sqrt[3]{2+x}$.
5. $(1+x)^{\frac{3}{2}}$.
6. $(1+x+x^2)^{\frac{1}{2}}$.

REVERSION OF SERIES.

490. To *revert* a series $y=ax+bx^2+cx^3+\cdots$ is to express x in a series of ascending powers of y.

Revert the series

$$y=x-2x^2+3x^3-4x^4+\cdots. \qquad (1).$$

Assume $$x=Ay+By^2+Cy^3+Dy^4+\cdots.$$

Substituting in this equation the value of y as given in (1), we have

$$\begin{aligned}x&=A(x-2x^2+3x^3-4x^4+\cdots)=A(x-2x^2+3x^3-4x^4+\cdots)\\&+B(x-2x^2+3x^3-4x^4+\cdots)^2=B(x^2+4x^4-4x^3+6x^4+\cdots)\\&+C(x-2x^2+3x^3-4x^4+\cdots)^3=C(x^3-6x^4-\cdots\qquad)\\&+D(x-2x^2+3x^3-4x^4+\cdots)^4=D(x^4+\cdots\qquad)\end{aligned}$$

Equating the coefficients of like powers of x,

$$A=1;\quad B-2A=0,\quad C-4B+3A=0,$$
$$\therefore B=2;\qquad \therefore C=5;$$
$$D-6C+10B-4A=0,$$
$$\therefore D=14.$$

Hence $$x=y+2y^2+5y^3+14y^4+\cdots.$$

491. If the series be $y = 1 + 2x + 3x^2 + 4x^3 + \cdots$ put $y - 1 = z$;

then $$z = 2x + 3x^2 + 4x^3 + \cdots.$$

Assume $x = Az + Bz^2 + Cz^3 + \cdots$ and complete the work as in Art. 490 after which replace z by its value $y - 1$.

EXAMPLES XLII. d.

Revert each of the following series to four terms:

1. $y = x + x^2 + x^3 + x^4 + \cdots.$
2. $y = x + 3x^2 + 5x^3 + 7x^4 + \cdots.$
3. $y = x - \frac{x^2}{2} + \frac{x^3}{4} - \frac{x^4}{8} + \cdots.$
4. $y = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots.$
5. $y = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots.$
6. $y = ax + bx^2 + cx^3 + dx^4 + \cdots.$

PARTIAL FRACTIONS.

492. A group of fractions connected by the signs of addition and subtraction is reduced to a more simple form by being collected into one single fraction whose denominator is the lowest common denominator of the given fractions. But the converse process of separating a fraction into a group of simpler, or **partial,** fractions is often required. For example, if we wish to expand $\frac{3 - 5x}{1 - 4x + 3x^2}$ in a series of ascending powers of x, we might use the method of Art. 487, and so obtain as many terms as we please. But if we wish to find the *general term* of the series, this method is inapplicable, and it is simpler to express the given fraction in the equivalent form $\frac{1}{1 - x} + \frac{2}{1 - 3x}$. Each of the expressions $(1 - x)^{-1}$ and $(1 - 3x)^{-1}$ can now be expanded by the Binomial Theorem, and the *general term* obtained.

493. We shall now give some examples illustrating the decomposition of a rational fraction into partial fractions. For a fuller discussion of the subject the reader is referred to treatises on Higher Algebra, or the Integral Calculus.

In these works it is proved that any rational fraction may be resolved into a series of partial fractions; and that

(1) To any factor of the first degree, as $x-a$, in the denominator there corresponds a partial fraction of the form $\dfrac{A}{x-a}\cdot$

(2) To any factor of the first degree, as $x-b$, occurring n times in the denominator there corresponds a series of n partial fractions of the form

$$\frac{B}{x-b}+\frac{C}{(x-b)^2}+\cdots+\frac{R}{(x-b)^n}.$$

(3) To any quadratic factors, as x^2+px+q, in the denominator there corresponds a partial fraction of the form $\dfrac{Ax+B}{x^2+px+q}.$

(4) To any quadratic factor, as x^2+px+q, occurring n times in the denominator there corresponds a series of n partial fractions of the form

$$\frac{Ax+B}{(x^2+px+q)}+\frac{Cx+D}{(x^2+px+q)^2}+\cdots+\frac{Rx+S}{(x^2+px+q)^n}.$$

Here the quantities A, B, C, D, $\cdots$ R, S, are all independent of x.

We shall make use of these results in the examples that follow.

Ex. 1. Separate $\dfrac{5x-11}{2x^2+x-6}$ into partial fractions.

Since the denominator $2x^2+x-6=(x+2)(2x-3)$, we assume

$$\frac{5x-11}{2x^2+x-6}=\frac{A}{x+2}+\frac{B}{2x-3},$$

where A and B are quantities independent of x whose values have to be determined.

Clearing of fractions,

$$5x-11=A(2x-3)+B(x+2).$$

Since this equation is identically true, we may equate coefficients of like powers of x; thus

$$2A+B=5,\quad -3A+2B=-11;$$

whence $$A=3,\ B=-1.$$

$$\therefore \frac{5x-11}{2x^2+x-6}=\frac{3}{x+2}-\frac{1}{2x-3}.$$

Ex. 2. Resolve $\dfrac{mx+n}{(x-a)(x+b)}$ into partial fractions.

Assume
$$\frac{mx+n}{(x-a)(x+b)}=\frac{A}{x-a}+\frac{B}{x+b}.$$
$$\therefore\ mx+n=A(x+b)+B(x-a) \quad . \quad . \quad . \quad . \quad (1).$$

We might now equate coefficients and find the values of A and B, but it is simpler to proceed in the following manner :

Since A and B are independent of x, we may give to x any value we please.

In (1) put $x-a=0$, or $x=a$; then
$$A=\frac{ma+n}{a+b};$$
putting $x+b=0$, or $x=-b$, $B=\dfrac{mb-n}{a+b}$.
$$\therefore\ \frac{mx+n}{(x-a)(x+b)}=\frac{1}{a+b}\left(\frac{ma+n}{x-a}+\frac{mb-n}{x+b}\right).$$

Ex. 3. Resolve $\dfrac{23x-11x^2}{(2x-1)(9-x^2)}$ into partial fractions.

Assume $\dfrac{23x-11x^2}{(2x-1)(3+x)(3-x)}=\dfrac{A}{2x-1}+\dfrac{B}{3+x}+\dfrac{C}{3-x}$. (1);
$$\therefore\ 23x-11x^2=A(3+x)(3-x)+B(2x-1)(3-x)$$
$$+C(2x-1)(3+x).$$

By equating the coefficients of like powers of x, or putting in succession $2x-1=0$, $3+x=0$, $3-x=0$, we find that
$$A=1,\ B=4,\ C=-1.$$
$$\therefore\ \frac{23x-11x^2}{(2x-1)(9-x^2)}=\frac{1}{2x-1}+\frac{4}{3+x}-\frac{1}{3-x}.$$

Ex. 4. Resolve $\dfrac{3x^2+x-2}{(x-2)^2(1-2x)}$ into partial fractions.

Assume $\dfrac{3x^2+x-2}{(x-2)^2(1-2x)}=\dfrac{A}{1-2x}+\dfrac{B}{x-2}+\dfrac{C}{(x-2)^2}$;
$$\therefore\ 3x^2+x-2=A(x-2)^2+B(1-2x)(x-2)+C(1-2x).$$

Let $1-2x=0$, then $\quad A=-\frac{1}{3}$;

let $x-2=0$, then $\quad C=-4$.

To find B, equate the coefficients of x^2; thus
$$3=A-2B;\ \text{whence}\ B=-\tfrac{5}{3}.$$
$$\therefore\ \frac{3x^2+x-2}{(x-2)^2(1-2x)}=-\frac{1}{3(1-2x)}-\frac{5}{3(x-2)}-\frac{4}{(x-2)^2}.$$

Ex. 5. Resolve $\dfrac{42 - 19x}{(x^2+1)(x-4)}$ into partial fractions.

Assume
$$\frac{42 - 19x}{(x^2+1)(x-4)} = \frac{Ax + B}{x^2+1} + \frac{C}{x-4};$$
$$\therefore\ 42 - 19x = (Ax + B)(x - 4) + C(x^2 + 1).$$

Let $x = 4$, then $\quad C = -2;$

equating coefficients of x^2, $\quad 0 = A + C$, and $A = 2$;

equating the absolute terms, $\quad 42 = -4B + C$, and $B = -11$,
$$\therefore\ \frac{42 - 19x}{(x^2+1)(x-4)} = \frac{2x - 11}{x^2+1} - \frac{2}{x-4}.$$

494. In all the preceding examples the numerator has been of lower dimensions than the denominator; if this is not the case, we divide the numerator by the denominator until a remainder is obtained which is of lower dimensions than the denominator.

Ex. Resolve $\dfrac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1}$ into partial fractions.

By division,
$$\frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = 2x + 3 + \frac{8x - 4}{3x^2 - 2x - 1};$$
and
$$\frac{8x - 4}{3x^2 - 2x - 1} = \frac{5}{3x+1} + \frac{1}{x-1};$$
$$\therefore\ \frac{6x^3 + 5x^2 - 7}{3x^2 - 2x - 1} = 2x + 3 + \frac{5}{3x+1} + \frac{1}{x-1}.$$

495. The General Term. We shall now explain how resolution into partial fractions may be used to facilitate the expansion of a rational fraction in ascending powers of x.

Ex. 1. Find the general term of $\dfrac{1 + 6x}{1 - 3x + 2x^2}$ when expanded in a series of ascending powers of x.

By Ex. 1, Art. 493, we have
$$\begin{aligned}\frac{1 + 5x}{1 - 3x + 2x^2} = \frac{7}{1-2x} - \frac{6}{1-x} &= 7(1 - 2x)^{-1} - 6(1 - x)^{-1} \\ &= 7[1 + (2x) + (2x)^2 + \cdots + (2x)^r + \cdots] \\ &\quad - 6(1 + x + x^2 + \cdots + x^r + \cdots).\end{aligned}$$

Hence the $(r + 1)$th, or general term of the expansion, is
$$7(2x)^r - 6x^r \text{ or } [7(2)^r - 6]x^r.$$

Ex. 2. Find the general term of $\dfrac{3x^2+x-2}{(x-2)^2(1-2x)}$ when expanded in a series of ascending powers of x.

By Ex. 4, Art. 493, we have

$$\frac{3x^2+x-2}{(x-2)^2(1-2x)}=-\frac{1}{3(1-2x)}-\frac{5}{3(x-2)}-\frac{4}{(x-2)^2}$$
$$=-\frac{1}{3(1-2x)}+\frac{5}{3(2-x)}-\frac{4}{(2-x)^2}$$
$$=-\frac{1}{3(1-2x)}+\frac{5}{6\left(1-\frac{x}{2}\right)}-\frac{4}{4\left(1-\frac{x}{2}\right)^2}$$
$$=-\frac{1}{3}(1-2x)^{-1}+\frac{5}{6}\left(1-\frac{x}{2}\right)^{-1}-\left(1-\frac{x}{2}\right)^{-2}$$
$$=-\tfrac{1}{3}[1+(2x)+(2x)^2+\cdots+(2x)^r+\cdots]$$
$$+\frac{5}{6}\left[1+\left(\frac{x}{2}\right)+\left(\frac{x}{2}\right)^2+\cdots+\left(\frac{x}{2}\right)^r+\cdots\right]$$
$$-\left[1+2\left(\frac{x}{2}\right)+3\left(\frac{x}{2}\right)^2+\cdots+(r+1)\left(\frac{x}{2}\right)^r+\cdots\right].$$

Hence the $(r+1)$th or general term of the expansion is

$$\left(-\frac{2^r}{3}+\frac{5}{6}\cdot\frac{1}{2^r}-\frac{r+1}{2^r}\right)x^r.$$

496. The following example sufficiently illustrates the method to be pursued when the denominator contains a quadratic factor.

Ex. Expand $\dfrac{7+x}{(1+x)(1+x^2)}$ in ascending powers of x and find the general term.

Assume $$\frac{7+x}{(1+x)(1+x^2)}=\frac{A}{1+x}+\frac{Bx+C}{1+x^2};$$

$$\therefore\ 7+x=A(1+x^2)+(Bx+C)(1+x).$$

Let $\quad 1+x=0$, then $A=3$;

equating the absolute terms, $\quad 7=A+C$, whence $C=4$;

equating the coefficients of x^2, $0=A+B$, whence $B=-3$.

$$\therefore\ \frac{7+x}{(1+x)(1+x^2)}=\frac{3}{1+x}+\frac{4-3x}{1+x^2}$$
$$=3(1+x)^{-1}+(4-3x)(1+x^2)^{-1}$$
$$=3\{1-x+x^2-\cdots+(-1)^px^p+\cdots\}$$
$$+(4-3x)\{1-x^2+x^4-\cdots+(-1)^px^{2p}+\cdots\}.$$

To find the coefficient of x^r;

(1) If r is even, the coefficient of x^r in the second series is $4(-1)^{\frac{r}{2}}$; therefore in the expansion the coefficient of x^r is $3+4(-1)^{\frac{r}{2}}$.

(2) If r is odd, the coefficient of x^r in the second series is $-3(-1)^{\frac{r-1}{2}}$, and the required coefficient is $3(-1)^{\frac{r+1}{2}}-3$.

EXAMPLES XLII. e.

Resolve into partial fractions:

1. $\dfrac{7x-1}{1-5x+6x^2}$.
2. $\dfrac{46+13x}{12x^2-11x-15}$.
3. $\dfrac{1+3x+2x^2}{(1-2x)(1-x^2)}$.
4. $\dfrac{x^2-10x+13}{(x-1)(x^2-5x+6)}$.
5. $\dfrac{2x^3+x^2-x-3}{x(x-1)(2x+3)}$.
6. $\dfrac{9}{(x-1)(x+2)^2}$.
7. $\dfrac{x^4-3x^3-3x^2+10}{(x+1)^2(x-3)}$.
8. $\dfrac{26x^2+208x}{(x^2+1)(x+5)}$.
9. $\dfrac{2x^2-11x+5}{(x-3)(x^2+2x-5)}$.
10. $\dfrac{3x^3-8x^2+10}{(x-1)^4}$.
11. $\dfrac{5x^3+6x^2+5x}{(x^2-1)(x+1)^3}$.

Find the general term of the following expressions when expanded in ascending powers of x.

12. $\dfrac{1+3x}{1+11x+28x^2}$.
13. $\dfrac{5x+6}{(2+x)(1-x)}$.
14. $\dfrac{x^2+7x+3}{x^2+7x+10}$.
15. $\dfrac{2x-4}{(1-x^2)(1-2x)}$.
16. $\dfrac{4+3x+2x^2}{(1-x)(1+x-2x^2)}$.
17. $\dfrac{3+2x-x^2}{(1+x)(1-4x)^2}$.
18. $\dfrac{4+7x}{(2+3x)(1+x)^2}$.
19. $\dfrac{2x+1}{(x-1)(x^2+1)}$.
20. $\dfrac{1-x+2x^2}{(1-x)^3}$.
21. $\dfrac{1}{(1-ax)(1-bx)(1-cx)}$.
22. $\dfrac{3-2x^2}{(2-3x+x^2)^2}$.

CHAPTER XLIII.

CONTINUED FRACTIONS.

497. An expression of the form $a + \cfrac{b}{c + \cfrac{d}{e + \cdots}}$ is called a **continued fraction**; here the letters $a, b, c, \cdots$ may denote any quantities whatever, but for the present we shall only consider the simpler form $a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}$, where $a_1, a_2, a_3, \cdots$ are positive integers. This will be usually written in the more compact form

$$a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +}\cdots.$$

498. When the number of *quotients* $a_1, a_2, a_3, \cdots$ is finite the continued fraction is said to be *terminating;* if the number of quotients is unlimited the fraction is called an *infinite continued fraction.*

It is possible to reduce every terminating continued fraction to an ordinary fraction by simplifying the fractions in succession beginning from the lowest.

499. To convert a given fraction into a continued fraction.

Let $\frac{m}{n}$ be the given fraction; divide m by n; let a_1 be the quotient and p the remainder; thus

$$\frac{m}{n} = a_1 + \frac{p}{n} = a_1 + \cfrac{1}{\frac{n}{p}};$$

divide n by p, let a_2 be the quotient and q the remainder; thus

$$\frac{n}{p} = a_2 + \frac{q}{p} = a_2 + \frac{1}{\frac{p}{q}};$$

divide p by q, let a_3 be the quotient and r the remainder; and so on. Thus

$$\frac{m}{n} = a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}} = a_1 + \frac{1}{a_2 +}\,\frac{1}{a_3 +} \cdots.$$

If m is less than n, the first quotient is zero, and we put

$$\frac{m}{n} = \frac{1}{\frac{n}{m}},$$

and proceed as before.

It will be observed that the above process is the same as that of finding the greatest common measure of m and n; hence if m and n are *commensurable*, we shall at length arrive at a stage where the division is exact and the process terminates. Thus every fraction whose numerator and denominator are positive integers can be converted into a terminating continued fraction.

Ex. Reduce $\frac{832}{159}$ to a continued fraction.

Find the greatest common measure of 832 and 159 by the usual process, thus:

$$\begin{array}{l}
159)832(5 \\
\quad\;\; \underline{795} \\
\quad\;\; 37)159(4 \\
\qquad\quad \underline{148} \\
\qquad\quad 11)37(3 \\
\qquad\qquad \underline{33} \\
\qquad\qquad 4)11(2 \\
\qquad\qquad\quad \underline{8} \\
\qquad\qquad\quad 3)4(1 \\
\qquad\qquad\qquad \underline{3} \\
\qquad\qquad\qquad 1)3(3 \\
\qquad\qquad\qquad\; \underline{3} \\
\qquad\qquad\qquad\; 0
\end{array}$$

We have the successive quotients 5, 4, 3, 2, 1, 3; hence

$$\frac{832}{159} = 5 + \frac{1}{4+}\,\frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3}.$$

500. Convergents. The fractions obtained by stopping at the first, second, third, ⋯ quotients of a continued fraction are called the first, second, third, ⋯ *convergents*, because, as will be shown in Art. 506, each successive convergent is a nearer approximation to the true value of the continued fraction than any of the preceding convergents.

501. To show that the convergents are alternately less and greater than the continued fraction.

Let the continued fraction be $a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\cdots$.

The first convergent is a_1, and is too small because the part $\frac{1}{a_2+}\,\frac{1}{a_3+}\cdots$ is omitted. The second convergent is $a_1 + \frac{1}{a_2}$, and is too great because the denominator a_2 is too small. The third convergent is $a_1 + \frac{1}{a_2+}\,\frac{1}{a_3}$, and is too small because $a_2 + \frac{1}{a_3}$ is too great; and so on.

When the given fraction is a proper fraction, $a_1 = 0$; if in this case we agree to consider zero as the first convergent, we may enunciate the above results as follows:

The convergents of an odd order are all less, and the convergents of an even order are all greater, than the continued fraction.

502. To establish the law of formation of the successive convergents.

Let the continued fraction be denoted by

$$a_1 + \frac{1}{a_2+}\,\frac{1}{a_3+}\,\frac{1}{a_4+}\cdots;$$

then the first three convergents are

$$\frac{a_1}{1},\quad \frac{a_1a_2+1}{a_2},\quad \frac{a_3(a_1a_2+1)+a_1}{a_3\cdot a_2+1};$$

and we see that the numerator of the third convergent may be formed by multiplying the numerator of the second convergent by the third quotient, and adding the numerator of the first convergent; also that the denominator may be formed in a similar manner.

Suppose that the successive convergents are formed in a similar way; let the numerators be denoted by $p_1, p_2, p_3, \cdots$, and the denominators by $q_1, q_2, q_3, \cdots$.

Assume that the law of formation holds for the nth convergent; that is, suppose

$$p_n = a_n p_{n-1} + p_{n-2},\quad q_n = a_n q_{n-1} + q_{n-2}.$$

The $(n+1)$th convergent differs from the nth only in having the quotient $a_n + \dfrac{1}{a_{n+1}}$ in the place of a_n; hence the $(n+1)$th convergent

$$= \frac{\left(a_n + \dfrac{1}{a_{n+1}}\right)p_{n-1} + p_{n-2}}{\left(a_n + \dfrac{1}{a_{n+1}}\right)q_{n-1} + q_{n-2}} = \frac{a_{n+1}(a_n p_{n-1} + p_{n-2}) + p_{n-1}}{a_{n+1}(a_n q_{n-1} + q_{n-2}) + q_{n-1}}$$

$$= \frac{a_{n+1}p_n + p_{n-1}}{a_{n+1}q_n + q_{n-1}}, \text{ by supposition.}$$

If therefore we put

$$p_{n+1} = a_{n+1}p_n + p_{n-1},\ q_{n+1} = a_{n+1}q_n + q_{n-1},$$

we see that the numerator and denominator of the $(n+1)$th convergent follow the law which was supposed to hold in the case of the nth. But the law does hold in the case of the third convergent, hence it holds for the fourth, and so on; therefore it holds universally.

Ex. Reduce $\frac{674}{313}$ to a continued fraction and calculate the successive convergents.

By Art. 499, $\dfrac{674}{313} = 2 + \dfrac{1}{6+}\,\dfrac{1}{1+}\,\dfrac{1}{1+}\,\dfrac{1}{11+}\,\dfrac{1}{2}.$

The successive quotients are 2, 6, 1, 1, 11, 2.

The successive convergents are $\frac{2}{1}, \frac{13}{6}, \frac{15}{7}, \frac{28}{13}, \frac{323}{150}, \frac{674}{313}$.

EXPLANATION. With the first and second quotients take the first and second convergents, which are readily determined. Thus, in this example, 2 is the first convergent, and $2 + \frac{1}{6}$ or $\frac{13}{6}$ the second convergent. The numerator of the third convergent, 15, equals the numerator of the preceding convergent, 13, multiplied by 1, the third quotient, plus 2, the numerator of the convergent next preceding but one. The denominator is formed in a similar manner: thus $7 = 1 \times 6 + 1$.

$$\text{The fifth convergent} = \frac{11(28)+15}{11(13)+7} = \frac{323}{150}.$$

503. If the fraction is a proper fraction, we may consider zero as the first convergent, and proceed as follows:

Reduce $\frac{84}{227}$ to a continued fraction, and calculate the successive convergents.

Proceeding as in Art. 499,

```
227)84(0
     00
     84)227(2
        168
         59)84(1
            59
            25)59(...
```

$$\text{We obtain}\quad 0 + \frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{2}.$$

The successive quotients are 0, 2, 1, 2, 2, 1, 3, 2.

Writing $\frac{0}{1}$ for the first convergent and arranging the work as show in the example of the preceding article, we have

Quotients	0	2	1	2	2	1	3	2.
Convergents	$\frac{0}{1}$,	$\frac{1}{2}$,	$\frac{1}{3}$,	$\frac{3}{8}$,	$\frac{7}{19}$,	$\frac{10}{27}$,	$\frac{37}{100}$,	$\frac{84}{227}$.

504. It will be convenient to call a_n the nth *partial* quotient; the *complete* quotient at this stage being

$$a_n + \frac{1}{a_{n+1}+}\,\frac{1}{a_{n+2}+}\cdots$$

We shall usually denote the complete quotient at any stage by k.

We have seen that

$$\frac{p_n}{q_n} = \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}}.$$

Let the continued fraction be denoted by x; then x differs from $\frac{p_n}{q_n}$ only in taking the complete quotient k instead of the partial quotient a_n; thus

$$x = \frac{k p_{n-1} + p_{n-2}}{k q_{n-1} + q_{n-2}}.$$

505. To show that if $\frac{p_n}{q_n}$ be the *n*th convergent to a continued fraction, then

$$p_n q_{n-1} - p_{n-1} q_n = (-1)_n.$$

Let the continued fraction be denoted by

$$a_1 + \frac{1}{a_2 +} \frac{1}{a_3 +} \frac{1}{a_4 +} \cdots;$$

then $p_n q_{n-1} - p_{n-1} q_n = (a_n p_{n-1} + p_{n-2}) q_{n-1} - p_{n-1}(a_n q_{n-1} + q_{n-2})$

$$= (-1)(p_{n-1} q_{n-2} - p_{n-2} q_{n-1})$$

$$= (-1)^2 p_{n-2} q_{n-3} - p_{n-3} q_{n-2}), \text{ similarly,}$$

$$= \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$

$$= (-1)^{n-2} (p_2 q_1 - p_1 q_2).$$

But $\quad p_2 q_1 - p_1 q_2 = (a_1 a_2 + 1) - a_1 \cdot a_2 = 1 = (-1)^2;$

hence $\quad p_n q_{n-1} - p_{n-1} q_n = (-1)^n.$

When the continued fraction is *less* than unity, this result will still hold if we suppose that $a_1 = 0$, and that the first convergent is zero.

NOTE. When we are calculating the numerical value of the successive convergents, the above theorem furnishes an easy test of the accuracy of the work.

COR. 1. *Each convergent is in its lowest terms;* for if p_n and q_n had a common divisor it would divide $p_n q_{n-1} - p_{n-1} q_n$, or unity; which is impossible.

Cor. 2. *The difference between two successive convergents is a fraction whose numerator is unity, and whose denominator is the product of the denominators of these convergents;* for

$$^{*}\frac{p_n}{q_n} \sim \frac{p_{n-1}}{q_{n-1}} = \frac{p_n q_{n-1} \sim p_{n-1} q_n}{q_n q_{n-1}} = \frac{1}{q_n q_{n-1}}.$$

506. Each convergent is nearer to the continued fraction than any of the preceding convergents.

Let x denote the continued fraction, and $\frac{p_n}{q_n}$, $\frac{p_{n+1}}{q_{n+1}}$, $\frac{p_{n+2}}{q_{n+2}}$ three consecutive convergents; then x differs from $\frac{p_{n+2}}{q_{n+2}}$ only in taking the *complete* $(n+2)$th quotient in the place of a_{n+2}; denote this by k; thus

$$x = \frac{kp_{n+1} + p_n}{kq_{n+1} + q_n};$$

$$\therefore\ x \sim \frac{p_n}{q_n} = \frac{k(p_{n+1} q_n \sim p_n q_{n+1})}{q_n(kq_{n+1} + q_n)} = \frac{k}{q_n(kq_{n+1} + q_n)},$$

and $$\frac{p_{n+1}}{q_{n+1}} \sim x = \frac{p_{n+1} q_n \sim p_n q_{n+1}}{q_{n+1}(kq_{n+1} + q_n)} = \frac{1}{q_{n+1}(kq_{n+1} + q_n)}.$$

Now k is greater than unity, and q_n is less than q_{n+1}; hence on both accounts the difference between $\frac{p_{n+1}}{q_{n+1}}$ and x is less than the difference between $\frac{p_n}{q_n}$ and x; that is, every convergent is nearer to the continued fraction than the next preceding convergent, and therefore nearer than any preceding convergent.

Combining the result of this article with that of Art. 501, it follows that

The convergents of an odd order continually increase, but are always less than the continued fraction;

The convergents of an even order continually decrease, but are always greater than the continued fraction.

* The sign $\sim$ means "difference between."

507. To find limits to the error made in taking any convergent for the continued fraction.

Let $\frac{p_n}{q_n}, \frac{p_{n+1}}{q_{n+1}}, \frac{p_{n+2}}{q_{n+2}}$ be three consecutive convergents, and let k denote the complete $(n+2)$th quotient;

then $$x = \frac{kp_{n+1} + p_n}{kq_{n+1} + q_n};$$

$$\therefore \; x \sim \frac{p_n}{q_n} = \frac{k}{q_n(kq_{n+1} + q_n)} = \frac{1}{q_n\left(q_{n+1} + \frac{q_n}{k}\right)}.$$

Now k is greater than 1, therefore *the difference between the continued fraction x, and any convergent, $\frac{p_n}{q_n}$, is less than $\frac{1}{q_n q_{n+1}}$, and greater than $\frac{1}{q_n(q_{n+1} + q_n)}$.*

Again, since $q_{n+1} > q_n$, the error in taking $\frac{p_n}{q_n}$ instead of x is less than $\frac{1}{q_n^{\,2}}$ and greater than $\frac{1}{2\,q^2_{\,n+1}}$.

508. From the last article it appears that the error in taking $\frac{p_n}{q_n}$ instead of the continued fraction is less than $\frac{1}{q_n q_{n+1}}$, or $\frac{1}{q_n(a_{n+1}q_n + q_{n-1})}$; that is, less than $\frac{1}{a_{n+1}q_n^{\,2}}$; hence the larger a_{n+1} is, the nearer does $\frac{p_n}{q_n}$ approximate to the continued fraction; therefore, *any convergent which immediately precedes a large quotient is a near approximation to the continued fraction.*

Again, since the error is less than $\frac{1}{q_n^{\,2}}$, it follows that in order to find a convergent which will differ from the continued fraction by less than a given quantity $\frac{1}{a}$, we have only to calculate the successive convergents up to $\frac{p_n}{q_n}$, where $q_n^{\,2}$ is greater than a.

509. The properties of continued fractions enable us to find two small integers whose ratio closely approximates to that of two incommensurable quantities, or to that of two quantities whose exact ratio can only be expressed by large integers.

Ex. Find a series of fractions approximating to 3.14159.

In the process of finding the greatest common measure of 14159 and 100000, the successive quotients are 7, 15, 1, 25, 1, 7, 4. Thus

$$3.14159 = 3 + \frac{1}{7+}\,\frac{1}{15+}\,\frac{1}{1+}\,\frac{1}{25+}\,\frac{1}{1+}\,\frac{1}{7+}\,\frac{1}{4}.$$

The successive convergents are

$$\tfrac{3}{1},\ \tfrac{22}{7},\ \tfrac{333}{106},\ \tfrac{355}{113},\ \ldots.$$

This last convergent which precedes the large quotient 25 is a very near approximation, the error being less than $\dfrac{1}{25 \times (113)^2}$, and therefore less than $\dfrac{1}{25 \times (100)^2}$, or .000004.

510. **Any convergent is nearer to the continued fraction than any other fraction whose denominator is less than that of the convergent.**

Let x be the continued fraction, $\dfrac{p_n}{q_n}$, $\dfrac{p_{n-1}}{q_{n-1}}$ two consecutive convergents, $\dfrac{r}{s}$ a fraction whose denominator s is less than q_n.

If possible, let $\dfrac{r}{s}$ be nearer to x than $\dfrac{p_n}{q_n}$, then $\dfrac{r}{s}$ must be nearer to x than $\dfrac{p_{n-1}}{q_{n-1}}$ [Art. 506]; and since x lies between $\dfrac{p_n}{q_n}$ and $\dfrac{p_{n-1}}{q_{n-1}}$, it follows that $\dfrac{r}{s}$ must lie between $\dfrac{p_n}{q_n}$ and $\dfrac{p_{n-1}}{q_{n-1}}$.

Hence

$$\frac{r}{s} \sim \frac{p_{n-1}}{q_{n-1}} < \frac{p_n}{q_n} \sim \frac{p_{n-1}}{q_{n-1}}, \text{ that is } < \frac{1}{q_n q_{n-1}};$$

$$\therefore\ rq_{n-1} \sim sp_{n-1} < \frac{s}{q_n};$$

that is, an integer less than a fraction; which is impossible. Therefore $\dfrac{p_n}{q_n}$ must be nearer to the continued fraction than $\dfrac{r}{s}$.

EXAMPLES XLIII. a.

Calculate the successive convergents to

1. $2+\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{2}.$

2. $\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{4+}\,\frac{1}{2+}\,\frac{1}{6}.$

3. $3+\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{9}.$

Express the following quantities as continued fractions and find the fourth convergent to each: also determine the limits to the error made by taking the third convergent for the fraction.

4. $\frac{253}{179}$. **6.** $\frac{1189}{3927}$. **8.** $.37$. **10.** $.3029$.

5. $\frac{251}{802}$. **7.** $\frac{729}{2318}$. **9.** 1.139. **11.** $4.31\dot{6}$.

12. Find limits to the error in taking $\frac{222}{203}$ yards as equivalent to a metre, given that a metre is equal to 1.0936 yards.

13. Find an approximation to

$$1+\frac{1}{3+}\,\frac{1}{5+}\,\frac{1}{7+}\,\frac{1}{9+}\,\frac{1}{11+}\cdots$$

which differs from the true value by less than .0001.

14. Show by the theory of continued fractions that $\frac{99}{70}$ differs from 1.41421 by a quantity less than $\frac{1}{11830}$.

RECURRING CONTINUED FRACTIONS.

511. We have seen that a *terminating* continued fraction with rational quotients can be reduced to an ordinary fraction with integral numerator and denominator, and therefore cannot be equal to a surd; but we shall prove that a quadratic surd can be expressed as an *infinite* continued fraction whose quotients recur. We shall first consider a numerical example.

Ex. Express $\sqrt{19}$ as a continued fraction, and find a series of fractions approximating to its value.

$$\sqrt{19}=4+(\sqrt{19}-4)=4+\frac{3}{\sqrt{19}+4};$$

$$\frac{\sqrt{19}+4}{3}=2+\frac{\sqrt{19}-2}{3}=2+\frac{5}{\sqrt{19}+2};$$

$$\frac{\sqrt{19}+2}{5}=1+\frac{\sqrt{19}-3}{5}=1+\frac{2}{\sqrt{19}+3};$$

$$\frac{\sqrt{19}+3}{2}=3+\frac{\sqrt{19}-3}{2}=3+\frac{5}{\sqrt{19}+3};$$

$$\frac{\sqrt{19}+3}{5}=1+\frac{\sqrt{19}-2}{5}=1+\frac{3}{\sqrt{19}+2};$$

$$\frac{\sqrt{19}+2}{3}=2+\frac{\sqrt{19}-4}{3}=2+\frac{1}{\sqrt{19}+4};$$

$$\sqrt{19}+4=8+(\sqrt{19}-4)=8+\cdots$$

after this the quotients 2, 1, 3, 1, 2, 8 recur; hence

$$\sqrt{19}=4+\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{3+}\,\frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{8+}\cdots.$$

It will be noticed that the quotients recur as soon as we come to a quotient which is double the first.

Explanation. We first find the greatest integer in $\sqrt{19}$; this is 4, and we write $\sqrt{19}=4+(\sqrt{19}-4)$. We then express $\sqrt{19}-4$ as an equivalent fraction with a rational *numerator*. Thus

$$\sqrt{19}-4=\frac{(\sqrt{19}-4)(\sqrt{19}+4)}{\sqrt{19}+4}=\frac{3}{\sqrt{19}+4}.$$

The work now stands

$$\sqrt{19}=4+\frac{3}{\sqrt{19}+4}=4+\frac{1}{\frac{\sqrt{19}+4}{3}}.$$

We begin the second line with $\frac{\sqrt{19}+4}{3}$, the denominator of this complex fraction, which is itself a fraction with a *rational denominator*. The greatest integer in this fraction is 2, and we write

$$\frac{\sqrt{19}+4}{3}=2+\frac{\sqrt{19}-2}{3}.$$

We then multiply numerator and denominator by the surd conjugate to $\sqrt{19}-2$, so that after inverting the result $\frac{5}{\sqrt{19}+2}$, we again begin a line with a rational denominator. The same series of operations is performed in each of the following lines.

The first seven convergents formed as explained in Art. 502 are

$$\frac{4}{1},\ \frac{9}{2},\ \frac{13}{3},\ \frac{48}{11},\ \frac{61}{14},\ \frac{170}{39},\ \frac{1421}{326}.$$

The error in taking the last of these is less than $\frac{1}{(326)^2}$, and is therefore less than $\frac{1}{(320)^2}$ or $\frac{1}{102400}$, and still less than .00001. Thus the seventh convergent gives the value to at least four places of decimals.

512. Every periodic continued fraction is equal to one of the roots of a quadratic equation of which the coefficients are rational.

Let x denote the continued fraction, and y the periodic part, and suppose that

$$x = a + \frac{1}{b+}\,\frac{1}{c+}\cdots\frac{1}{h+}\,\frac{1}{k+}\,\frac{1}{y},$$

and

$$y = m + \frac{1}{n+}\cdots\frac{1}{u+}\,\frac{1}{v+}\,\frac{1}{y},$$

where $a, b, c, \cdots h, k, m, n, \cdots u, v$ are positive integers.

Let $\frac{p}{q}, \frac{p'}{q'}$ be the convergents to x corresponding to the quotients h, k respectively; then since y is the complete quotient, we have $x = \frac{p'y + p}{q'y + q}$; whence $y = \frac{p - qx}{q'x - p'}$.

Let $\frac{r}{s}, \frac{r'}{s'}$ be the convergents to y corresponding to the quotients u, v respectively; then $y = \frac{r'y + r}{s'y + s}$.

Substituting for y in terms of x and simplifying, we obtain a quadratic of which the coefficients are rational.

The equation $s'y^2 + (s - r')y - r = 0$, which gives the value of y, has its roots real and of opposite signs; if the positive value of y be substituted in $x = \frac{p'y + p}{q'y + q}$, on rationalizing the denominator the value of x is of the form $\frac{A + \sqrt{B}}{C}$, where A, B, C are integers, B being positive since the value of y is real.

Ex. Express $1 + \frac{1}{2+}\;\frac{1}{3+}\;\frac{1}{2+}\;\frac{1}{3+}\cdots$ as a surd.

Let x be the value of the continued fraction; then

$$x - 1 = \frac{1}{2+}\;\frac{1}{3 + (x-1)};$$

whence $2x^2 + 2x - 7 = 0$.

The continued fraction is equal to the positive root of this equation, and is therefore equal to $\frac{\sqrt{15} - 1}{2}$.

EXAMPLES XLIII. b.

Express the following surds as continued fractions, and find the sixth convergent to each:

1. $\sqrt{3}$. **2.** $\sqrt{5}$. **3.** $\sqrt{6}$. **4.** $\sqrt{8}$. **5.** $\sqrt{11}$.

6. $\sqrt{13}$. **7.** $\sqrt{14}$. **8.** $\sqrt{22}$. **9.** $2\sqrt{3}$. **10.** $4\sqrt{2}$.

11. $3\sqrt{5}$. **12.** $4\sqrt{10}$. **13.** $\frac{1}{\sqrt{21}}$.

14. $\frac{1}{\sqrt{33}}$. **15.** $\sqrt{\frac{6}{5}}$. **16.** $\sqrt{\frac{7}{11}}$.

17. Find limits of the error when $\frac{268}{65}$ is taken for $\sqrt{17}$.

18. Find limits of the error when $\frac{916}{191}$ is taken for $\sqrt{23}$.

19. Find the first convergent to $\sqrt{101}$ that is correct to five places of decimals.

20. Find the first convergent to $\sqrt{15}$ that is correct to five places of decimals.

Express as a continued fraction the positive root of each of the following equations:

21. $x^2+2x-1=0$. **22.** $x^2-4x-3=0$. **23.** $7x^2-8x-3=0$.

24. Express each root of $x^2 - 5x + 3 = 0$ as a continued fraction.

25. Find the value of $3 + \frac{1}{6+}\;\frac{1}{6+}\;\frac{1}{6+}\;\cdots$.

26. Find the value of $\frac{1}{1+}\;\frac{1}{3+}\;\frac{1}{1+}\;\frac{1}{3+}\;\cdots$.

27. Find the value of $3 + \frac{1}{1+}\;\frac{1}{2+}\;\frac{1}{3+}\;\frac{1}{1+}\;\frac{1}{2+}\;\frac{1}{3+}\;\cdots$.

28. Find the value of $5 + \frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{1+}\;\frac{1}{10+}\;\cdots$.

CHAPTER XLIV.

SUMMATION OF SERIES.

513. Examples of the summation of certain series (Arithmetic and Geometric) have occurred in previous chapters. We will now consider methods for summing other series.

514. Recurring Series. A series $u_0 + u_1 + u_2 + u_3 + \cdots$, in which from and after a certain term each term is equal to the sum of a fixed number of the preceding terms multiplied respectively by certain constants, is called a *recurring series*. A recurring series is of the 1st, 2d, or rth order, according as 1, 2, or r constants are required as multipliers.

515. Scale of Relation. In the series

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots,$$

each term after the second is equal to the sum of the two preceding terms multiplied respectively by the *constants* $2x$ and $-x^2$; these quantities being called constants because they are the same for all values of n. Thus

$$5x^4 = 2x \cdot 4x^3 + (-x^2) \cdot 3x^2;$$

that is,

$$u_4 = 2xu_3 - x^2u_2;$$

and generally, when n is greater than 1, each term is connected with the two that immediately precede it by the equation

$$u_n = 2xu_{n-1} - x^2u_{n-2},$$

or,

$$u_n - 2xu_{n-1} + x^2u_{n-2} = 0.$$

In this equation the coefficients of u_n, u_{n-1}, and u_{n-2}, *taken with their proper signs*, form what is called the *scale of relation*.

Thus the series

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots$$

is a recurring series in which the scale of relation is

$$1 - 2x + x^2.$$

516. To find any term when the scale of relation is given. If the scale of relation of a recurring series is given, any term can be found when a sufficient number of the preceding terms are known. As the method of procedure is the same, however many terms the scale of relation may consist of, the following illustration will be sufficient:

If $$1 - px - qx^2 - rx^3$$

is the scale of relation of the series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

we have

$$a_nx^n = px \cdot a_{n-1}x^{n-1} + qx^2 \cdot a_{n-2}x^{n-2} + rx^3 \cdot a_{n-3}x^{n-3},$$

or $$a_n = pa_{n-1} + qa_{n-2} + ra_{n-3};$$

thus any coefficient can be found when the coefficients of the three preceding terms are known.

517. To find the scale of relation. If a sufficient number of the terms of a series be given, the scale of relation may be found.

Ex. Find the scale of relation of the recurring series

$$2 + 5x + 13x^2 + 35x^3 + 97x^4 + 275x^5 + 793x^6 + \cdots.$$

This is plainly not a series of the first order. If it be of the second order, to obtain p and q we have the equations

$$13 = 5p + 2q, \text{ and } 35 = 13p + 5q;$$

whence $p = 5$, and $q = -6$. By using these values of p and q, we can obtain the fifth and sixth coefficients; hence they are correct, and the scale of relation is

$$1 - 5x + 6x^2.$$

If we could not have obtained the remaining coefficients with these values of p and q, we would have assumed the series to be of the third order, and formed the equations

$$35 = 13p + 5q + 2r,$$
$$97 = 35p + 13q + 5r,$$
$$275 = 97p + 35q + 13r;$$

whence values for p, q, and r would have been obtained, and trial with the seventh and following coefficients would have shown whether they were correct.

518. If the scale of relation consists of 3 terms it involves 2 constants, p and q; and we must have 2 equations to determine p and q. To obtain the first of these we must know at least 3 terms of the series, and to obtain the second we must have one more term given. Thus to obtain a scale of relation involving two constants we must have at least 4 terms given.

If the scale of relation be $1 - px - qx^2 - rx^3$, to find the 3 constants we must have 3 equations. To obtain the first of these we must know at least 4 terms of the series, and to obtain the other two we must have two more terms given; hence to find a scale of relation involving 3 constants, at least 6 terms of the series must be given.

Generally, to find a scale of relation involving m constants, we must know at least $2m$ consecutive terms.

Conversely, if $2m$ consecutive terms are given, we may assume for the scale of relation

$$1 - p_1x - p_2x^2 - p_3x^3 - \cdots - p_mx^m.$$

519. The Sum of n Terms of a Recurring Series. The method of finding the sum is the same whatever be the scale of relation; for simplicity we shall suppose it to contain only two constants.

Let the series be

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

and let the sum be S; let the scale of relation be $1-px-qx^2$; so that for every value of n greater than 1, we have

$$a_n - pa_{n-1} - qa_{n-2} = 0.$$

Now $S = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$,

$$-pxS = -pa_0x - pa_1x^2 - \cdots - pa_{n-2}x^{n-1} - pa_{n-1}x^n,$$

$$-qx^2S = -qa_0x^2 - \cdots - qa_{n-3}x^{n-1} - qa_{n-2}x^n - qa_{n-1}x^{n+1}.$$

Hence

$$(1-px-qx^2)S=a_0+(a_1-pa_0)x-(pa_{n-1}+qa_{n-2})x^n-qa_{n-1}x^{n+1},$$

for the coefficient of every other power of x is zero in consequence of the relation

$$a_n-pa_{n-1}-qa_{n-2}=0.$$

$$\therefore\ S=\frac{a_0+(a_1-pa_0)x}{1-px-qx^2}-\frac{(pa_{n-1}+qa_{n-2})x^n+qa_{n-1}x^{n+1}}{1-px-qx^2}.$$

Thus the sum of a recurring series is a fraction whose denominator is the scale of relation.

520. If the second fraction in the result of the last article decreases indefinitely as n increases indefinitely, the formula for the sum of an *infinite* number of terms of a recurring series of the *second order* reduces to

$$S=\frac{a_0+(a_1-pa_0)x}{1-px-qx^2}.$$

If we develop this fraction in ascending powers of x as explained in Art. 487, we shall obtain as many terms of the original series as we please; for this reason the expression

$$\frac{a_0+(a_1-pa_0)x}{1-px-qx^2}$$

is called the *generating function* * of the series. The **summation** of the series is the finding of this generating function.

If the series is of the *third order*,

$$S=\frac{a_0+(a_1-pa_0)x+(a_2-pa_1-qa_0)x^2}{1-px-qx^2-rx^3}.$$

521. From the result of Art. 519, we obtain

$$\frac{a_0+(a_1-pa_0)x}{1-px-qx^2}=a_0+a_1x+a_2x^2+\cdots+a_{n-1}x^{n+1}$$
$$+\frac{(pa_{n-1}+qa_{n-2})x^n+qa_{n-1}x^{n+1}}{1-px-qx^2};$$

* Sometimes called the *generating fraction*.

from which we see that although the generating function

$$\frac{a_0 + (a_1 - pa_0)x}{1 - px - qx^2}$$

may be used to obtain as many terms of the series as we please, it can be regarded as the true equivalent to the infinite series

$$a_0 + a_1x + a_2x^2 + \cdots,$$

only if the remainder

$$\frac{(pa_{n-1} + qa_{n-2})x^n + qa_{n-1}x^{n+1}}{1 - px - qx^2}$$

vanishes when n is indefinitely increased; in other words only when the series is convergent.

522. The General Term. When the *generating function* can be expressed as a group of partial fractions the general term of a recurring series may be easily found.

Ex. Find the generating function, and the general term, of the recurring series

$$1 - 7x - x^2 - 43x^3 - \cdots$$

Let the scale of relation be $1 - px - qx^2$; then

$$-1 + 7p - q = 0, \quad -43 + p + 7q = 0;$$

whence $p = 1$, $q = 6$; and the scale of relation is

$$1 - x - 6x^2.$$

Let S denote the sum of the series; then

$$\begin{aligned} S &= 1 - 7x - x^2 - 43x^3 - \cdots \\ -xS &= - x + 7x^2 + x^3 + \cdots \\ -6x^2S &= - 6x^2 + 42x^3 + \cdots \end{aligned}$$

$$\therefore (1 - x - 6x^2)S = 1 - 8x,$$

$$S = \frac{1 - 8x}{1 - x - 6x^2};$$

which is the generating function.

If we separate $\frac{1 - 8x}{1 - x - 6x^2}$ into partial fractions, we obtain

$$\frac{2}{1 + 2x} - \frac{1}{1 - 3x}.$$

By actual division, or by the Binomial Theorem.

$$\frac{2}{1+2x} = 2[1 - 2x + (2x)^2 - \cdots + (-1)^r(2x)^r + \cdots]$$

$$-\frac{1}{1-3x} = -[1 + 3x + (3x)^2 + \cdots + (3x)^r + \cdots]$$

Whence the $(r+1)$th, or general term, is

$$[2(2^r)(-1)^r - 3^r]\,x^r = \{(-1)^r 2^{r+1} - 3^r\}x^r.$$

EXAMPLES XLIV. a.

Find the generating functions of the following series.

1. $1 + 6x + 24x^2 + 84x^3 + \cdots$.

2. $2 + 2x - 2x^2 + 6x^3 - 14x^4 + \cdots$.

3. $3 - 16x + 42x^2 - 94x^3 + \cdots$.

4. $2 - 5x + 4x^2 + 7x^3 - 26x^4 + \cdots$.

5. $4 + 5x + 7x^2 + 11x^3 + \cdots$.

6. $1 + x + 2x^2 + 2x^3 + 3x^4 + 3x^5 + 4x^6 + 4x^7 + \cdots$.

7. $1 + 3x + 7x^2 + 13x^3 + 21x^4 + 31x^5 + \cdots$.

8. $1 - 3x + 5x^2 - 7x^3 + 9x^4 - 11x^5 + \cdots$.

Find the generating function and the general term in each of the following series:

9. $1 + 5x + 9x^2 + 13x^3 + \cdots$.

10. $2 - x + 5x^2 - 7x^3 + \cdots$.

11. $2 + 3x + 5x^2 + 9x^3 + \cdots$.

12. $7 - 6x + 9x^2 + 27x^4 + \cdots$.

13. $3 + 6x + 14x^2 + 36x^3 + 98x^4 + 276x^5 + \cdots$.

THE METHOD OF DIFFERENCES.

523. Let u_n denote some rational integral function of n, and let $u_1, u_2, u_3, u_4, \cdots$ denote the values of u_n when for n the values 1, 2, 3, 4, $\cdots$ are written successively.

From the series $u_1, u_2, u_3, u_4, u_5, \cdots$ obtain a second series by subtracting each term from the term which immediately follows it.

The series $u_2 - u_1,\ u_3 - u_2,\ u_4 - u_3,\ u_5 - u_4, \cdots$ thus found is called the *series of the first order of differences*, and may be conveniently denoted by $Du_1, Du_2, Du_3, Du_4, \cdots$.

By subtracting each term of this series from the term that immediately follows it, we have $Du_2 - Du_1,\ Du_3 - Du_2,$

$Du_4 - Du_3$, ... which may be called the *series of the second order of differences*, and denoted by D_2u_1, D_2u_2, D_2u_3, ...

From this series we may proceed to form the *series of the third, fourth, fifth, ⋯ orders of differences*, the general terms of these series being D_3u_r, D_4u_r, D_5u_r, ⋯ respectively.

524. Any Required Term of the Series. From the law of formation of the series

$$\begin{array}{c} u_1, \quad u_2, \quad u_3, \quad u_4, \quad u_5, \quad u_6, \cdots \\ Du_1, \; Du_2, \; Du_3, \; Du_4, \; Du_5, \cdots \\ D_2u_1, \; D_2u_2, \; D_2u_3, \; D_2u_4, \cdots \\ D_3u_1, \; D_3u_2, \; D_3u_3, \cdots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{array}$$

it appears that any term in any series is equal to the term immediately preceding it added to the term below it on the left.

Thus $\quad u_2 = u_1 + Du_1$, and $Du_2 = Du_1 + D_2u_1$.

By addition, since $u_2 + Du_2 = u_3$, we have

$$u_3 = u_1 + 2\,Du_1 + D_2u_1.$$

In an exactly similar manner by using the second, third, and fourth series in place of the first, second, and third, we obtain $Du_3 = Du_1 + 2\,D_2u_1 + D_3u_1$.

By addition, since $u_3 + Du_3 = u_4$, we have

$$u_4 = u_1 + 3\,Du_1 + 3\,D_2u_1 + D_3u_1.$$

So far as we have proceeded, the numerical coefficients follow the same law as those of the Binomial Theorem. We shall now prove by induction that this will always be the case. For suppose that

$$u_{n+1} = u_1 + nDu_1 + \frac{n(n-1)}{1 \cdot 2}D_2u_1 + \cdots + {}^nC_rD_ru_1 + \cdots + D_nu_1;$$

then by using the second to the $(n+2)$th series in the place of the first to the $(n+1)$th series we have

$$Du_{n+1} = Du_1 + nD_2u_1 + \frac{n(n-1)}{1 \cdot 2}D_3u_1 + \cdots + {}^nC_{r-1}D_ru_1 + \cdots + D_{n+1}u_1.$$

By addition, since $u_{n+1} + Du_{n+1} = u_{n+2}$, we obtain

$$u_{n+2} = u_1 + (n+1)Du_1 + \cdots + ({}^nC_r + {}^nC_{r-1})D_r u_1 + \cdots + D_{n+1}u_1.$$

But $${}^nC_r + {}^nC_{r-1} = \left(\frac{n-r+1}{r} + 1\right) \times {}^nC_{r-1} = \frac{n+1}{r} \times {}^nC_{r-1}$$

$$= \frac{(n+1)n(n-1)\cdots(\overline{n+1}-r+1)}{1 \cdot 2 \cdot 3 \cdots (r-1)r} = {}^{n+1}C_r.$$

Hence if the law of formation holds for u_{n+1} it also holds for u_{n+2}, but it is true in the case of u_4, therefore it holds for u_5, and therefore universally. Hence

$$u_n = u_1 + (n-1)Du_1 + \frac{(n-1)(n-2)}{1 \cdot 2} D_2 u_1 + \cdots + D_{n-1}u_1.$$

If we take a as the first term of a given series, $d_1, d_2, d_3 \ldots$ as the *first terms* of the successive orders of differences, any term of the given series is obtained from the formula

$$a_n = a + (n-1)\,d_1 + \frac{(n-1)(n-2)}{\underline{|2}} d_2$$
$$+ \frac{(n-1)(n-2)(n-3)}{\underline{|3}} d_3 + \cdots.$$

525. The Sum of *n* Terms of the Series. Suppose the series $u_1, u_2, u_3, \cdots$ is the first order of differences of the series

$$v_1, \quad v_2, \quad v_3, \quad v_4, \cdots,$$

then $v_{n+1} = (v_{n+1} - v_n) + (v_n - v_{n-1}) + \cdots + (v_2 - v_1) + v_1$, identically;

$$\therefore\ v_{n+1} = u_n + u_{n-1} + \cdots + u_2 + u_1 + v_1.$$

Hence in the series

$$\begin{array}{ccccc} 0, & v_2, & v_3, & v_4, & v_5 \cdots \\ & u_1, & u_2, & u_3, & u_4 \cdots \\ & Du_1, & Du_2, & Du_3 \cdots & \end{array}$$

the law of formation is the same as in the preceding article;

$$\therefore\ v_{n+1} = 0 + nu_1 + \frac{n(n-1)}{1 \cdot 2} Du_1 + \cdots + D_n u_1;$$

that is, $u_1 + u_2 + u_3 + \cdots + u_n$

$$= nu_1 + \frac{n(n-1)}{\lfloor 2} Du_1 + \frac{n(n-1)(n-2)}{\lfloor 3} D_2u_1 + \cdots + D_nu_1.$$

If, as in the preceding article, a is the first term of a given series, d_1, d_2, d_3, $\cdots$ *the first terms* of the successive orders of differences, the sum of n terms of the given series is obtained from the formula

$$S_n = na + \frac{n(n-1)}{\lfloor 2} d_1 + \frac{n(n-1)(n-2)}{\lfloor 3} d_2 + \frac{n(n-1)(n-2)(n-3)}{\lfloor 4} d_3 + \cdots.$$

Ex. 1. Find the 7th term and the sum of the first seven terms of the series 4, 14, 30, 52, 80, $\cdots$.

The successive orders of differences are

$$\begin{array}{c} 10, \quad 16, \quad 22, \quad 28, \\ 6, \quad 6, \quad 6, \\ 0, \quad 0. \end{array}$$

Here $n = 7$, and $a = 4$.

Hence, using formula, Art. 524, the 7th term

$$= 4 + 6 \cdot 10 + \frac{6 \cdot 5}{1 \cdot 2} \cdot 6 = 154.$$

Using formula, Art. 525, the sum of the first seven terms

$$= 7 \cdot 4 + \frac{7 \cdot 6}{1 \cdot 2} \cdot 10 + \frac{7 \cdot 6 \cdot 5}{1 \cdot 2 \cdot 3} \cdot 6 = 448.$$

Ex. 2. Find the general term and the sum of n terms of the series

$$12, \quad 40, \quad 90, \quad 168, \quad 280, \quad 432, \cdots.$$

The successive orders of difference are

$$\begin{array}{c} 28, \quad 50, \quad 78, \quad 112, \quad 152, \cdots \\ 22, \quad 28, \quad 34, \quad 40, \cdots \\ 6, \quad 6, \quad 6, \cdots \\ 0, \quad 0, \cdots \end{array}$$

Hence the nth term [Art. 524]

$$= 12 + 28(n-1) + \frac{22(n-1)(n-2)}{\lfloor 2} + \frac{6(n-1)(n-2)(n-3)}{\lfloor 3}$$

$$= n^3 + 5n^2 + 6n.$$

Using the formula for the sum of n terms we obtain

$$S_n = 12n + \frac{28n(n-1)}{\lfloor 2} + \frac{22n(n-1)(n-2)}{\lfloor 3} + \frac{6n(n-1)(n-2)(n-3)}{\lfloor 4}$$

$$= \frac{n}{12}(3n^2 + 26n + 69n + 46)$$

$$= \tfrac{1}{12}n(n+1)(3n^2 + 23n + 46).$$

526. It will be seen that this method of summation will only succeed when the series is such that in forming the orders of differences we eventually come to a series in which all the terms are equal. This will always be the case if the nth term of the series is a rational integral function of n.

PILES OF SHOT AND SHELLS.

527. Square Pile. *To find the number of shot arranged in a complete pyramid on a square base.*

The top layer consists of a single shot; the next contains 4; the next 9, and so on to n^2, n being the number of layers: hence the form of the series is

	1^2,		2^2,		3^2,		$4^2, \cdots, n^2$.
Series	1,		4,		9,		$16, \cdots, n^2$.
1st order of differences		3,		5,		7,	
2d order of differences			2,		2,		
3d order of differences				0.			

Substituting in Art. 525, we obtain

$$S = n + \frac{n(n-1)}{1 \cdot 2} \cdot 3 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} \cdot 2 = \frac{n(n+1)(2n+1)}{6}.$$

528. Triangular Pile. *To find the number of shot arranged in a complete pyramid the base of which is an equilateral triangle.*

The top layer consists of a single shot; the next contains 3; the next 6; the next 10, and so on, giving a series of the form

	1,	1+2,	1+2+3,	1+2+3+4, ⋯
Series	1,	3,	6,	10,
1st order of differences	2,	3,	4,	
2d order of differences	1,	1,		
3d order of differences	0.			

Hence

$$S = n + \frac{n(n-1)}{1\cdot 2}\cdot 2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\cdot 1 = \frac{n(n+1)(n+2)}{6}.$$

529. Rectangular Pile. *To find the number of shot arranged in a complete pile the base of which is a rectangle.*

The top layer consists of a single row of shot. Suppose this row to contain m shot; then the next layer contains $2(m+1)$; the next $3(m+2)$, and so on, giving a series of the form

	m,	$2m+2$,	$3m+6$,	$4m+12, \cdots$
1st order of differences	$m+2$,	$m+4$,	$m+6$,	
2d order of differences	2	2,		
3d order of differences	0.			

Now let l and w be the number of shot in the length and width, respectively, of the base; then $m = l - w + 1$.

Making these substitutions, we have

$$S = \frac{n(n+1)(3l - w + 1)}{6}.$$

EXAMPLES XLIV. b.

1. Find the eighth term and the sum of the first eight terms of the series 1, 8, 27, 64, 125, ⋯.

2. Find the tenth term and the sum of the first ten terms of the series 4, 11, 28, 55, 92, ⋯.

Find the number of shot in :

3. A square pile, having 15 shot in each side of the base.

4. A triangular pile, having 18 shot in each side of the base.

5. A rectangular pile, the length and the breadth of the base containing 50 and 28 shot respectively.

6. An incomplete triangular pile, a side of the base having 25 shot, and a side of the top 14.

7. An incomplete square pile of 27 courses, having 40 shot in each side of the base.

8. Find the ninth term and the sum of the first nine terms of the series $1, 3+5, 7+9+11, \cdots$.

The numbers $1, 2, 3, \cdots$ are often referred to as the *natural numbers*.

9. Find the sum of the squares of the first n natural numbers.

10. Find the sum of the cubes of the first n natural numbers.

11. The number of shot in a complete rectangular pile is 24395; if there are 34 shot in the breadth of the base, how many are there in its length?

12. The number of shot in the top layer of a square pile is 169, and in the lowest layer is 1089; how many shot does the pile contain?

13. Find the number of shot in a complete rectangular pile of 15 courses, having 20 shot in the longer side of its base.

14. Find the number of shot in an incomplete rectangular pile, the number of shot in the sides of its upper course being 11 and 18, and the number in the shorter side of its lowest course being 30.

Find the nth term and the sum of n terms of the series:

15. $4, 14, 30, 52, 80, 114, \cdots$.

16. $8, 26, 54, 92, 140, 198, \cdots$.

17. $2, 12, 36, 80, 150, 252, \cdots$.

18. $8, 16, 0, -64, -200, -432, \cdots$.

19. $30, 144, 420, 960, 1890, 3360, \cdots$.

20. What is the number of shot required to complete a rectangular pile having 15 and 6 shot in the longer and shorter side, respectively, of its upper course?

21. The number of shot in a triangular pile is greater by 150 than half the number of shot in a square pile, the number of layers in each being the same: find the number of shot in the lowest layer of the triangular pile.

22. Find the number of shot in an incomplete square pile of 16 courses when the number of shot in the upper course is 1005 less than in the lowest course.

23. Show that the number of shot in a square pile is one-fourth the number of shot in a triangular pile of double the number of courses.

INTERPOLATION.

530. The process of introducing between the terms of a series intermediate values conforming to the law of the series is called **interpolation**. An important application is in finding numbers intermediate between those given in logarithmic and other mathematical tables. For this purpose we may employ the formula used in finding the nth term by the Differential Method, giving fractional values to n.

Ex. Given $\log 40 = 1.6021$, $\log 41 = 1.6128$, $\log 42 = 1.6232$, $\log 43 = 1.6335$, $\cdots$ find $\log 40.7$.

Series	1.6021,		1.6128,		1.6232,		1.6335,
1st order of differences,		.0107,		.0104,		.0103,	
2d order of differences,			$-.0003$,		$-.0001$,		
3d order of differences,				$+.0002$.			

Substituting in formula of Art. 524, we have

$$\log 40.7 = 1.6021 + \frac{7}{10}(.0107) + \frac{7}{10}\left(-\frac{3}{10}\right)\left(\frac{-.0003}{\underline{|2}}\right)$$
$$+\frac{7}{10}\left(-\frac{3}{10}\right)\left(-\frac{13}{10}\right)\left(\frac{.0002}{\underline{|3}}\right)$$
$$= 1.6021 + .00749 + .000031 + .000009 = 1.6096 +.$$

Here $\log 40$ is the first term $(n=1)$; $\log 41$ is the second term $(n=2)$; hence in introducing the intermediate term $\log 40.7$ we give to n a value 1.7.

EXAMPLES XLIV. c.

1. Given $\log 3 = 0.4771$, $\log 4 = 0.6021$, $\log 5 = 0.6990$, $\log 6 = 0.7782$, $\cdots$; find $\log 4.4$.

2. Given $\log 51 = 1.7076$, $\log 52 = 1.7160$, $\log 53 = 1.7243$, $\log 54 = 1.7324$, $\cdots$; find $\log 51.9$.

3. Given $\sqrt{5} = 2.236$, $\sqrt{6} = 2.449$, $\sqrt{7} = 2.645$, $\sqrt{8} = 2.828$; find $\sqrt{5.6}$, $\sqrt{7.4}$, and $\sqrt{7.74}$.

4. Given $\sqrt[3]{51} = 3.7084$, $\sqrt[3]{52} = 3.7325$, $\sqrt[3]{53} = 3.7563$, $\cdots$; find $\sqrt[3]{51.18}$.

CHAPTER XLV.

Binomial Theorem. Any Index.

531. In Chapter XXXVII. we investigated the Binomial Theorem when the index was any positive integer; we shall now consider whether the formulæ there obtained hold in the case of negative and fractional values of the index.

Since, by Art. 411, every binomial may be reduced to one common type, it will be sufficient to confine our attention to binomials of the form $(1+x)^n$.

By actual evolution we have

$$(1+x)^{\frac{1}{2}} = \sqrt{1+x} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 - \cdots;$$

and by actual division,

$$(1-x)^{-2} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots;$$

and in each of these series the number of terms is unlimited.

In these cases we have by independent processes obtained an expansion for each of the expressions $(1+x)^{\frac{1}{2}}$ and $(1+x)^{-2}$. We shall presently prove that they are only particular cases of the general formula for the expansion of $(1+x)^n$, where n is any rational quantity.

This formula was discovered by Newton.

532. Suppose we have two expressions arranged in ascending powers of x, such as

$$1 + mx + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots \qquad (1),$$

and $$1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots \qquad (2).$$

The product of these two expressions will be a series in ascending powers of x; denote it by

$$1 + Ax + Bx^2 + Cx^3 + Dx^4 + \cdots;$$

then it is clear that $A, B, C, \cdots$ are functions of m and n, and therefore the actual values of $A, B, C, \cdots$ in any particular case will depend upon the values of m and n in that case. But the way in which the coefficients of the powers of x in (1) and (2) combine to give $A, B, C, \cdots$ is quite independent of m and n; in other words, *whatever values m and n may have $A, B, C, \cdots$ preserve the same invariable form.* If therefore we can determine the form of $A, B, C, \cdots$ for any value of m and n, we conclude that $A, B, C, \cdots$ will have the same form *for all values* of m and n.

The principle here explained is often referred to as an example of "the permanence of equivalent forms"; in the present case we have only to recognize the fact that *in any algebraic product* the *form* of the result will be the same whether the quantities involved are whole numbers, or fractions; positive, or negative.

We shall make use of this principle in the general proof of the Binomial Theorem for any index. The proof which we give is due to Euler.

533. To prove the Binomial Theorem when the index is a positive fraction.

Whatever be the value of m, positive or negative, integral or fractional, let the symbol $f(m)$ stand for the series

$$1 + mx + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots;$$

then $f(n)$ will stand for the series

$$1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \cdots.$$

If we multiply these two series together the product will be another series in ascending powers of x, whose *coefficients will be unaltered in form whatever m and n may be.*

To determine this *invariable form of the product* we may give to m and n any values that are most convenient; for this purpose suppose that m and n are positive integers. In this case $f(m)$ is the expanded form of $(1+x)^m$, and $f(n)$ is the expanded form of $(1+x)^n$; and therefore

$$f(m)\times f(n)=(1+x)^m\times(1+x)^n=(1+x)^{m+n},$$

but when m and n are positive integers, the expansion of $(1+x)^{m+n}$ is

$$1+(m+n)x+\frac{(m+n)(m+n-1)}{1\cdot 2}x^2+\cdots.$$

This then is the *form* of the product of $f(m)\times f(n)$ *in all cases,* whatever the values of m and n may be; and in agreement with our previous notation, it may be denoted by $f(m+n)$; therefore *for all values of m and n*

$$f(m)\times f(n)=f(m+n).$$

Also $\quad f(m)\times f(n)\times f(p)=f(m+n)\times f(p)$

$$=f(m+n+p),\text{ similarly.}$$

Proceeding in this way we may show that

$f(m)\times f(n)\times f(p)\cdots$ to k factors $=f(m+n+p+\cdots$ to k terms$)$.

Let each of these quantities, m, n, p, $\cdots$, be equal to $\frac{h}{k}$, where h and k are positive integers;

$$\therefore\ \left\{f\left(\frac{h}{k}\right)\right\}^k=f(h);$$

but since h is a positive integer, $f(h)=(1+x)^h$;

$$\therefore\ (1+x)^h=\left\{f\left(\frac{h}{k}\right)\right\}^k;$$

$$\therefore\ (1+x)^{\frac{h}{k}}=f\left(\frac{h}{k}\right);$$

but $f\left(\frac{h}{k}\right)$ stands for the series

$$1+\frac{h}{k}x+\frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{1\cdot 2}x^2+\cdots;$$

$$\therefore\ (1+x)^{\frac{h}{k}} = 1 + \frac{h}{k}x + \frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{1\cdot 2}x^2 + \cdots,$$

which proves the Binomial Theorem for any positive fractional index.

534. To prove the Binomial Theorem when the index is any negative quantity.

It has been proved that

$$f(m)\times f(n) = f(m+n)$$

for all values of m and n. Replacing m by $-n$ (where n is positive), we have

$$f(-n)\times f(n) = f(-n+n) = f(0) = 1,$$

since all terms of the series except the first vanish;

$$\therefore\ \frac{1}{f(n)} = f(-n);$$

but $f(n) = (1+x)^n$ for *any positive* value of n;

$$\therefore\ \frac{1}{(1+x)^n} = f(-n),$$

or

$$(1+x)^{-n} = f(-n).$$

But $f(-n)$ stands for the series

$$1 + (-n)x + \frac{(-n)(-n-1)}{1\cdot 2}x^2 + \cdots;$$

$$\therefore\ (1+x)^{-n} = 1 + (-n)x + \frac{(-n)(-n-1)}{1\cdot 2}x^2 + \cdots;$$

which proves the Binomial Theorem for any negative index.

535. It should be noticed that when $x < 1$, each of the series $f(m)$, $f(n)$, $f(m+n)$ is *convergent*, and $f(m+n)$ is the true arithmetical equivalent of $f(m)\times f(n)$. But when $x > 1$, all these series are *divergent*, and we can only assert that if we multiply the series denoted by $f(m)$, by the series denoted by $f(n)$, the first r terms of the product will agree with the first r terms of $f(m+n)$, whatever finite value r may have.

CHAPTER XLVI.

Exponential and Logarithmic Series.

536. The advantages of common logarithms have been explained in Art. 438, and in practice no other system is used. But in the first place these logarithms are calculated to another base and then transformed to base 10.

In the present chapter we shall prove certain formulæ, known as the **Exponential and Logarithmic Series,** and give a brief explanation of the way in which they are used in constructing a table of logarithms.

537. To expand a^x in ascending powers of x.

By the Binomial Theorem, if $n > 1$.

$$\left(1+\frac{1}{n}\right)^{xn}$$

$$=1+nx\cdot\frac{1}{n}+\frac{nx(nx-1)}{\lfloor 2}\cdot\frac{1}{n^2}+\frac{nx(nx-1)(nx-2)}{\lfloor 3}\cdot\frac{1}{n^3}+\cdots$$

$$=1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}+\cdots \quad . \quad . \quad . \quad (1).$$

By putting $x=1$, we obtain

$$\left(1+\frac{1}{n}\right)^n=1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\cdots \quad (2).$$

But $$\left(1+\frac{1}{n}\right)^{nx}=\left\{\left(1+\frac{1}{n}\right)^n\right\}^x;$$

hence the series (1) is the xth power of the series (2); that is,

$$1+x+\frac{x\left(x-\frac{1}{n}\right)}{\lfloor 2}+\frac{x\left(x-\frac{1}{n}\right)\left(x-\frac{2}{n}\right)}{\lfloor 3}+\cdots$$

$$=\left\{1+1+\frac{1-\frac{1}{n}}{\lfloor 2}+\frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)}{\lfloor 3}+\cdots\right\}^x;$$

and this is true however great n may be. If, therefore, n be indefinitely increased, we have

$$1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\cdots=\left(1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots\right)^x.$$

The series $$1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\cdots$$

is usually denoted by e; hence

$$e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\frac{x^4}{\lfloor 4}+\cdots$$

Write cx for x, then

$$e^{cx}=1+cx+\frac{c^2x^2}{\lfloor 2}+\frac{c^3x^3}{\lfloor 3}+\cdots.$$

Now let $e^c=a$, so that $c=\log_e a$; by substituting for c we obtain

$$a^x=1+x\log_e a+\frac{x^2(\log_e a)^2}{\lfloor 2}+\frac{x^3(\log_e a)^3}{\lfloor 3}+\cdots.$$

This is the **Exponential Theorem.**

538. The series

$$1+1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\cdots,$$

which we have denoted by e, is very important, as it is the base to which logarithms are first calculated. Logarithms to this base are known as the Napierian system, so named after Napier, the inventor of logarithms. They are also called *natural* logarithms from the fact that they are the first logarithms which naturally come into consideration in algebraic investigations.

When logarithms are used in theoretical work it is to be remembered that the base e is always understood, just as in arithmetical work the base 10 is invariably employed.

From the series the approximate value of e can be determined to any required degree of accuracy; to 10 places of decimals it is found to be 2.7182818284.

Ex. 1. Find the sum of the infinite series

$$1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\cdots.$$

We have $$e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\cdots;$$

and by putting $x=-1$ in the series for e^x, we obtain

$$e^{-1}=1-1+\frac{1}{\underline{|2}}-\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}-\cdots.$$

$$\therefore\ e+e^{-1}=2\left(1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|4}}+\frac{1}{\underline{|6}}+\cdots\right);$$

hence the sum of the series is $\frac{1}{2}(e+e^{-1})$.

Ex. 2. Find the coefficient of x^r in the expansion of $\dfrac{a-bx}{e^x}$.

$$\frac{a-bx}{e^x}=(a-bx)e^{-x}$$

$$=(a-bx)\left\{1-x+\frac{x^2}{\underline{|2}}-\frac{x^3}{\underline{|3}}+\cdots+\frac{(-1)^r x^r}{\underline{|r}}+\cdots\right\}.$$

The coefficient required $$=\frac{(-1)^r}{\underline{|r}}\cdot a-\frac{(-1)^{r-1}}{\underline{|r-1}}\cdot b$$

$$=\frac{(-1)^r}{\underline{|r}}(a+rb).$$

539. To expand $\log_e(1+x)$ in ascending powers of x.

From Art. 537,

$$a^y=1+y\log_e a+\frac{y^2(\log_e a)^2}{\underline{|2}}+\frac{y^3(\log_e a)^3}{\underline{|3}}+\cdots.$$

In this series write $1 + x$ for a; thus $(1 + x)^y$

$$=1+y\log_e(1+x)+\frac{y^2}{\underline{|2}}\{\log_e(1+x)\}^2+\frac{y^3}{\underline{|3}}\{\log_e(1+x)\}^3+\cdots(1).$$

Also by the Binomial Theorem, when $y<1$ we have

$$(1+x)^y=1+yx+\frac{y(y-1)}{\underline{|2}}x^2+\frac{y(y-1)(y-2)}{\underline{|3}}x^3+\cdots \quad . \quad (2).$$

Now in (2) the coefficient of y is

$$x+\frac{(-1)}{1\cdot 2}x^2+\frac{(-1)(-2)}{1\cdot 2\cdot 3}x^3+\frac{(-1)(-2)(-3)}{1\cdot 2\cdot 3\cdot 4}x^4+\cdots;$$

that is, $$x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots.$$

Equate this to the coefficient of y in (1); thus we have

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots.$$

This is known as the **Logarithmic Series.**

540. Except when x is very small the series for $\log_e(1+x)$ is of little use for numerical calculations. We can, however, deduce from it other series by the aid of which Tables of Logarithms may be constructed.

541. In Art. 539 we have proved that

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots;$$

changing x into $-x$, we have

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\cdots.$$

By subtraction,

$$\log_e\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\right).$$

Put $\frac{1+x}{1-x}=\frac{n+1}{n}$, so that $x=\frac{1}{2n+1}$; we thus obtain

$$\log_e(n+1)-\log_e n=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\cdots\right\}.$$

From this formula by putting $n = 1$ we can obtain $\log_e 2$. Again by putting $n = 2$ we obtain $\log_e 3 - \log_e 2$; whence $\log_e 3$ is found, and therefore also $\log_e 9$ is known.

Now by putting $n = 9$ we obtain $\log_e 10 - \log_e 9$; thus the value of $\log_e 10$ is found to be $2.30258509 \cdots$.

To convert Napierian logarithms into logarithms to base 10 we multiply by $\dfrac{1}{\log_e 10}$, which is the *modulus* [Art. 441] of the common system, and its value is $\dfrac{1}{2.30258509 \cdots}$, or $.43429448 \cdots$; we shall denote this modulus by M.

By multiplying the last series throughout by M we obtain a formula adapted to the calculation of common logarithms. Thus

$$M \log_e(n+1) - M \log_e n =$$
$$2M\left\{\frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots\right\};$$

that is, $\log_{10}(n+1) - \log_{10} n =$

$$2\left\{\frac{M}{2n+1} + \frac{M}{3(2n+1)^3} + \frac{M}{5(2n+1)^5} + \cdots\right\}.$$

Hence if the logarithm of one of two consecutive numbers be known, the logarithm of the other may be found, and thus a table of logarithms can be constructed.

EXAMPLES XLVI.

1. Show that

$$(1)\quad e^{-2} = 1 - \frac{2^3}{\underline{|3}} + \frac{2^4}{\underline{|4}} - \frac{2^5}{\underline{|5}} + \cdots;$$

$$(2)\quad \frac{e^2 - 1}{2e} = 1 + \frac{1}{\underline{|3}} + \frac{1}{\underline{|5}} + \frac{1}{\underline{|7}} + \cdots.$$

2. Expand $\log \sqrt{1+x}$ in ascending powers of x.

3. Prove that $\log_e 2 = \frac{1}{2} + \frac{1}{12} + \frac{1}{30} + \frac{1}{56} + \cdots$.

4. Show that $\log_{10}\left(\dfrac{1}{1-x}\right) = \dfrac{1}{\log_e 10}\left(x + \dfrac{x^2}{2} + \dfrac{x^3}{3} + \cdots\right)$.

5. Prove that $\log \dfrac{1+x}{1-3x} = 4x + 4x^2 + \dfrac{28}{3}x^3 + 20x^4 + \cdots$.

6. Show that if $x > 1$, $\log \sqrt{x^2 - 1} = \log x - \dfrac{1}{2x^2} - \dfrac{1}{4x^4} - \dfrac{1}{6x^6} - \cdots$.

CHAPTER XLVII.

DETERMINANTS.

542. Consider two homogeneous linear equations

$$a_1x + b_1y = 0,$$
$$a_2x + b_2y = 0;$$

multiplying the first equation by b_2, the second by b_1, subtracting and dividing by x, we obtain

$$a_1b_2 - a_2b_1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

This result is sometimes written

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0,$$

and the expression on the left is called a **determinant**. It consists of two *rows* and two *columns*, and in its expanded form or *development*, as seen in the first member of (1), each term is the product of two quantities; it is therefore said to be of the *second order*. The line a_1b_2 is called the *principal diagonal*, and the line b_1a_2, the *secondary diagonal*.

The letters a_1, b_1, a_2, b_2 are called the *constituents* of the determinant, and the terms a_1b_2, a_2b_1 are called the *elements*.

THE VALUE OF THE DETERMINANT AFTER CERTAIN CHANGES.

543. Since $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1 = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$,

it follows that *the value of the determinant is not altered by changing the rows into columns, and the columns into rows.*

Again, it is easily seen that

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix}, \text{ and } \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix};$$

that is, *if we interchange two rows or two columns of the determinant, we obtain a determinant which differs from it only in sign.*

544. Let us now consider the homogeneous linear equations

$$a_1x + b_1y + c_1z = 0,$$
$$a_2x + b_2y + c_2z = 0,$$
$$a_3x + b_3y + c_3z = 0.$$

By eliminating x, y, z, we obtain

$$a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2) = 0,$$

or
$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + b_1\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} = 0.$$

This is usually written

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

and the expression on the left being a determinant which consists of three rows and three columns is called a determinant of *the third order.*

545. By a rearrangement of terms, the expanded form of the above determinant may be written

$$a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1),$$

or
$$a_1\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} + a_2\begin{vmatrix} b_3 & b_1 \\ c_3 & c_1 \end{vmatrix} + a_3\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix};$$

hence
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix};$$

that is, *the value of the determinant is not altered by changing the rows into columns, and the columns into rows.*

546. Minors. From the preceding article,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} + a_2 \begin{vmatrix} b_3 & c_3 \\ b_1 & c_1 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \quad . \quad (1).$$

Also from Art. 544,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix} \quad . \quad . \quad (2).$$

We shall now explain a simple method of writing down the expansion of a determinant of the third order, and it should be noticed that it is immaterial whether we develop it from the first row or the first column.

From equation (1) we see that the coefficient of any one of the constituents a_1, a_2, a_3 is that determinant of the second order which is obtained by *omitting* the row and column in which it occurs. These determinants are called the *minors* of the original determinant, and the left-hand side of equation (1) may be written

$$a_1A_1 - a_2A_2 + a_3A_3,$$

where A_1, A_2, A_3 are the minors of a_1, a_2, a_3 respectively.

Again, from equation (2), the determinant is equal to

$$a_1A_1 - b_1B_1 + c_1C_1,$$

where A_1, B_1, C_1 are the minors of a_1, b_1, c_1 respectively.

547. The determinant $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

$$= a_1(b_2c_3 - b_3c_2) + b_1(c_2a_3 - c_3a_2) + c_1(a_2b_3 - a_3b_2)$$

$$= - b_1(a_2c_3 - a_3c_2) - a_1(c_2b_3 - c_3b_2) - c_1(b_2a_3 - b_3a_2);$$

hence $$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix}.$$

Thus it appears that *if two adjacent columns, or rows, of the determinant are interchanged, the sign of the determinant is changed, but its value remains unaltered.*

If for the sake of brevity we denote the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

by $(a_1b_2c_3)$, then the result we have just obtained may be written

$$(b_1a_2c_3) = -(a_1b_2c_3).$$

Similarly we may show that

$$(c_1a_2b_3) = -(a_1c_2b_3) = +(a_1b_2c_3).$$

548. Vanishing of a Determinant. *If two rows or two columns of the determinant are identical the determinant vanishes.*

For let D be the value of the determinant, then by interchanging two rows or two columns we obtain a determinant whose value is $-D$; but the determinant is unaltered; hence $D = -D$, that is $D = 0$. Thus we have the following equations,

$$a_1A_1 - a_2A_2 + a_3A_3 = D,$$
$$b_1A_1 - b_2A_2 + b_3A_3 = 0,$$
$$c_1A_1 - c_2A_2 + c_3A_3 = 0.$$

549. Multiplication of a Determinant. *If each constituent in any row, or in any column, is multiplied by the same factor, then the determinant is multiplied by that factor.*

For
$$\begin{vmatrix} ma_1 & b_1 & c_1 \\ ma_2 & b_2 & c_2 \\ ma_3 & b_3 & c_3 \end{vmatrix}$$

$$= ma_1 \cdot A_1 - ma_2 \cdot A_2 + ma_3 \cdot A_3 = m(a_1A_1 - a_2A_2 + a_3A_3);$$

which proves the proposition.

Cor. If each constituent of one row, or column, is the same multiple of the corresponding constituent of another row, or column, the determinant vanishes.

550. A Determinant expressed as the Sum of Two Other Determinants. *If each constituent in any row, or column, consists of two terms, then the determinant can be expressed as the sum of two other determinants.*

Thus we have

$$\begin{vmatrix} a_1+d_1 & b_1 & c_1 \\ a_2+d_2 & b_2 & c_2 \\ a_3+d_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix};$$

for the expression on the left,

$$=(a_1+d_1)A_1-(a_2+d_2)A_2+(a_3+d_3)A_3$$
$$=(a_1A_1-a_2A_2+a_3A_3)+(d_1A_1-d_2A_2+d_3A_3);$$

which proves the proposition.

In like manner if each constituent in any one row, or column, consists of m terms, the determinant can be expressed as the sum of m other determinants.

Similarly, we may show that

$$\begin{vmatrix} a_1+d_1 & b_1+e_1 & c_1 \\ a_2+d_2 & b_2+e_2 & c_2 \\ a_3+d_3 & b_3+e_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & e_1 & c_1 \\ a_2 & e_2 & c_2 \\ a_3 & e_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} d_1 & e_1 & c_1 \\ d_2 & e_2 & c_2 \\ d_3 & e_3 & c_3 \end{vmatrix}.$$

In general if the constituents of the three columns consist of m, n, p terms, respectively, the determinant can be expressed as the sum of mnp determinants.

Ex. 1. Show that $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix} = 3abc-a^3-b^3-c^3.$

The given determinant

$$= \begin{vmatrix} b & a & a \\ c & b & b \\ a & c & c \end{vmatrix} - \begin{vmatrix} b & b & a \\ c & c & b \\ a & a & c \end{vmatrix} + \begin{vmatrix} c & a & a \\ a & b & b \\ b & c & c \end{vmatrix} - \begin{vmatrix} c & b & a \\ a & c & b \\ b & a & c \end{vmatrix}.$$

Of these four determinants the first three vanish, Art. 548; thus the expression reduces to the last of the four determinants; hence its value

$$= -\{c(c^2 - ab) - b(ac - b^2) + a(a^2 - bc)\}$$
$$= 3abc - a^3 - b^3 - c^3.$$

Ex. 2. Find the value of $\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix}$.

We have

$$\begin{vmatrix} 67 & 19 & 21 \\ 39 & 13 & 14 \\ 81 & 24 & 26 \end{vmatrix} = \begin{vmatrix} 10+57 & 19 & 21 \\ 0+39 & 13 & 14 \\ 9+72 & 24 & 26 \end{vmatrix} = \begin{vmatrix} 10 & 19 & 21 \\ 0 & 13 & 14 \\ 9 & 24 & 26 \end{vmatrix} + \begin{vmatrix} 57 & 19 & 21 \\ 39 & 13 & 14 \\ 72 & 24 & 26 \end{vmatrix}$$

$$= \begin{vmatrix} 10 & 19 & 21 \\ 0 & 13 & 14 \\ 9 & 24 & 26 \end{vmatrix} = \begin{vmatrix} 10 & 19 & 19+2 \\ 0 & 13 & 13+1 \\ 9 & 24 & 24+2 \end{vmatrix} = \begin{vmatrix} 10 & 19 & 2 \\ 0 & 13 & 1 \\ 9 & 24 & 2 \end{vmatrix}$$

$$= 10\begin{vmatrix} 13 & 1 \\ 24 & 2 \end{vmatrix} + 9\begin{vmatrix} 19 & 2 \\ 13 & 1 \end{vmatrix} = 20 - 63 = -43.$$

551. Simplification of Determinants. Consider the determinant

$$\begin{vmatrix} a_1 + pb_1 + qc_1 & b_1 & c_1 \\ a_2 + pb_2 + qc_2 & b_2 & c_2 \\ a_3 + pb_3 + qc_3 & b_3 & c_3 \end{vmatrix};$$

as in the last article we can show that it is equal to

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pb_1 & b_1 & c_1 \\ pb_2 & b_2 & c_2 \\ pb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} qc_1 & b_1 & c_1 \\ qc_2 & b_2 & c_2 \\ qc_3 & b_3 & c_3 \end{vmatrix};$$

and the last two of these determinants vanish [Art. 549, Cor.]. Thus we see that the given determinant is equal to a new one whose first column is obtained by subtracting from the constituents of the first column of the original determinant equimultiples of the corresponding constituents of the other columns, while the second and third columns remain unaltered.

Conversely,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 + pb_1 + qc_1 & b_1 & c_1 \\ a_2 + pb_2 + qc_2 & b_2 & c_2 \\ a_3 + pb_3 + qc_3 & b_3 & c_3 \end{vmatrix};$$

and what has been here proved with reference to the first column is equally true for any of the columns or rows; hence it appears that in reducing a determinant we may replace any one of the rows or columns by a new row or column formed in the following way:

Take the constituents of the row or column to be replaced, and increase or diminish them by any equimultiples of the corresponding constituents of one or more of the other rows or columns.

After a little practice it will be found that determinants may often be quickly simplified by replacing two or more rows or columns simultaneously: for example, it is easy to see that

$$\begin{vmatrix} a_1+pb_1 & b_1-qc_1 & c_1 \\ a_2+pb_2 & b_2-qc_2 & c_2 \\ a_3+pb_3 & b_3-qc_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix};$$

but in any modification of the rule as above enunciated, care must be taken to leave one row or column unaltered.

Thus, if on the left-hand side of the last identity the constituents of the third column were replaced by c_1+ra_1, c_2+ra_2, c_3+ra_3, respectively, we should have the former value increased by

$$\begin{vmatrix} a_1+pb_1 & b_1-qc_1 & ra_1 \\ a_2+pb_2 & b_2-qc_2 & ra_2 \\ a_3+pb_3 & b_3-qc_3 & ra_3 \end{vmatrix},$$

and of the four determinants into which this may be resolved there is one which does not vanish, namely

$$\begin{vmatrix} pb_1 & -qc_1 & ra_1 \\ pb_2 & -qc_2 & ra_2 \\ pb_3 & -qc_3 & ra_3 \end{vmatrix}.$$

Ex. 1. Find the value of $\begin{vmatrix} 29 & 26 & 22 \\ 25 & 31 & 27 \\ 63 & 54 & 46 \end{vmatrix}$.

The given determinant

$$= \begin{vmatrix} 3 & 26 & -4 \\ -6 & 31 & -4 \\ 9 & 54 & -8 \end{vmatrix} = -3 \times 4 \times \begin{vmatrix} 1 & 26 & 1 \\ -2 & 31 & 1 \\ 3 & 54 & 2 \end{vmatrix} = -12 \times \begin{vmatrix} 1 & 26 & 1 \\ -3 & 5 & 0 \\ 1 & 2 & 0 \end{vmatrix}$$

$$= -12 \begin{vmatrix} 1 & 1 & 26 \\ 0 & -3 & 5 \\ 0 & 1 & 2 \end{vmatrix} = -12 \begin{vmatrix} -3 & 5 \\ 1 & 2 \end{vmatrix} = 132.$$

Explanation. In the first step of the reduction keep the second column unaltered; for the first new column diminish each constituent of the first column by the corresponding constituent of the second; for the third new column diminish each constituent of the third column by the corresponding constituent of the second. In the second step take out the factors 3 and -4. In the third step keep the first row unaltered; for the second new row diminish the constituents of the second by the corresponding ones of the first; for the third new row diminish the constituents of the third by twice the corresponding constituents of the first. The remaining steps will be easily seen.

Ex. 2. Show that $\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3.$

The given determinant

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c) \times \begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c) \times \begin{vmatrix} 1 & 0 & 0 \\ 2b & -b-c-a & 0 \\ 2c & 0 & -c-a-b \end{vmatrix}$$

$$= (a+b+c) \times \begin{vmatrix} -b-c-a & 0 \\ 0 & -c-a-b \end{vmatrix} = (a+b+c)^3.$$

Explanation. In the first new determinant the first row is the sum of the constituents of the three rows of the original determinant, the second and third rows being unaltered. In the third of the new determinants the first column remains unaltered, while the second and third columns are obtained by subtracting the constituents of the first column from those of the second and third respectively. The remaining transformations are sufficiently obvious.

EXAMPLES XLVII. a.

Calculate the values of the determinants:

1. $\begin{vmatrix} 1 & 1 & 1 \\ 35 & 37 & 34 \\ 23 & 26 & 25 \end{vmatrix}$.

2. $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$.

3. $\begin{vmatrix} 13 & 3 & 23 \\ 30 & 7 & 53 \\ 39 & 9 & 70 \end{vmatrix}$.

4. $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

5. $\begin{vmatrix} 1 & z & -y \\ -z & 1 & x \\ y & -x & 1 \end{vmatrix}$.

6. $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$.

7. $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$.

8. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$.

9. Without expanding the determinants, prove that

$$\begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = \begin{vmatrix} y & b & q \\ x & a & p \\ z & c & r \end{vmatrix} = \begin{vmatrix} x & y & z \\ p & q & r \\ a & b & c \end{vmatrix}.$$

Solve the equations:

10. $\begin{vmatrix} a & a & x \\ m & m & m \\ b & x & b \end{vmatrix} = 0.$

11. $\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0.$

Prove the following identities:

12. $\begin{vmatrix} b+c & c+a & a+b \\ q+r & r+p & p+q \\ y+z & z+x & x+y \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$.

13. $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = (b-c)(c-a)(a-b).$

14. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (b-c)(c-a)(a-b)(a+b+c).$

Calculate the value of the determinants:

15. $\begin{vmatrix} 3 & 6 & 9 \\ 4 & 7 & 10 \\ 5 & 8 & 11 \end{vmatrix}$.

16. $\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$.

APPLICATION TO THE SOLUTION OF SIMULTANEOUS EQUATIONS OF THE FIRST DEGREE.

552. The properties of determinants may be usefully employed in solving simultaneous linear equations.

Let the equations be

$$a_1x + b_1y + c_1z + d_1 = 0,$$
$$a_2x + b_2y + c_2z + d_2 = 0,$$
$$a_3x + b_3y + c_3z + d_3 = 0;$$

multiply them by A_1, $-A_2$, A_3 respectively and add the results, A_1, A_2, A_3 being minors of a_1, a_2, a_3 in the determinant

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

The coefficients of y and z vanish in virtue of the relations proved in Art. 548, and we obtain

$$(a_1A_1 - a_2A_2 + a_3A_3)x + (d_1A_1 - d_2A_2 + d_3A_3) = 0.$$

Similarly we may show that

$$(b_1B_1 - b_2B_2 + b_3B_3)y + (d_1B_1 - d_2B_2 + d_3B_3) = 0,$$

and $\quad (c_1C_1 - c_2C_2 + c_3C_3)z + (d_1C_1 - d_2C_2 + d_3C_3) = 0.$

Now $\quad a_1A_1 - a_2A_2 + a_3A_3 = -(b_1B_1 - b_2B_2 + b_3B_3)$

$$= c_1C_1 - c_2C_2 + c_3C_3 = D;$$

hence the solution may be written

$$\frac{x}{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} d_1 & a_1 & c_1 \\ d_2 & a_2 & c_2 \\ d_3 & a_3 & c_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} d_1 & a_1 & b_1 \\ d_2 & a_2 & b_2 \\ d_3 & a_3 & b_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

or more symmetrically

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Ex. Solve

$$x+2y+3z-13=0,$$
$$2x+y+z-7=0,$$
$$3x+4y+3z-21=0.$$

We have

$$\frac{x}{\begin{vmatrix} 2 & 3 & -13 \\ 1 & 1 & -7 \\ 4 & 3 & -21 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} 1 & 3 & -13 \\ 2 & 1 & -7 \\ 3 & 3 & -21 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 2 & -13 \\ 2 & 1 & -7 \\ 3 & 4 & -21 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 4 & 3 \end{vmatrix}}$$

or

$$\frac{x}{-8}=\frac{-y}{24}=\frac{z}{-16}=\frac{-1}{8};$$

whence $x=1$, $y=3$, and $z=2$.

Explanation. The denominator of x is a determinant formed by taking the coefficients in each column *except that of* x. In the same manner for the denominators of y and z we omit the columns of coefficients of y and z. The last determinant is formed by taking the three columns of coefficients of x, y, and z.

553. Suppose we have the system of four homogeneous linear equations

$$a_1x+b_1y+c_1z+d_1u=0,$$
$$a_2x+b_2y+c_2z+d_2u=0,$$
$$a_3x+b_3y+c_3z+d_3u=0,$$
$$a_4x+b_4y+c_4z+d_4u=0.$$

From the last three of these, we have, as in the preceding article,

$$\frac{x}{\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix}} = \frac{-u}{\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix}}.$$

Substituting in the first equation, the eliminant is

$$a_1\begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \\ a_4 & c_4 & d_4 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \\ a_4 & b_4 & d_4 \end{vmatrix} - d_1\begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{vmatrix} = 0.$$

This may be more concisely written in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0;$$

the expression on the left being a determinant of the fourth order.

Also we see that the coefficients of a_1, b_1, c_1, d_1 taken with their proper signs are the *minors* obtained by omitting the row and column which respectively contain these constituents.

554. More generally, if we have n homogeneous linear equations

$$\begin{aligned} a_1x_1 + b_1x_2 + c_1x_3 + \cdots + k_1x_n &= 0, \\ a_2x_1 + b_2x_2 + c_2x_3 + \cdots + k_2x_n &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad & \cdot \quad \cdot \quad \cdot \\ a_nx_1 + b_nx_2 + c_nx_3 + \cdots + k_nx_n &= 0, \end{aligned}$$

involving n unknown quantities x_1, x_2, x_3, $\cdots$ x_n, these quantities can be eliminated and the result expressed in the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \cdots k_1 \\ a_2 & b_2 & c_2 \cdots k_2 \\ \cdot & \cdot & \cdot \quad \cdot \quad \cdot \\ a_n & b_n & c_n \cdots k_n \end{vmatrix} = 0.$$

The left-hand member of this equation is a determinant which consists of n rows and n columns, and is called a determinant of the nth order.

The discussion of this more general form of determinant is beyond the scope of the present work; it will be sufficient here to remark that the properties which have been established in the case of determinants of the second and third orders are quite general, and are capable of being extended to determinants of any order.

555. Signs of the Terms. Although we may always develop a determinant by means of the process described above, it is not always the simplest method, especially when our object is not so much to find the value of the whole determinant, as to find the signs of its several elements.

556. The expanded form of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1;$$

and it appears that each element is the product of three factors, one taken from each row, and one from each column; also the signs of half the terms are $+$ and of the other half $-$. When written as above the signs of the several elements may be obtained as follows. The first element $a_1b_2c_3$, in which the suffixes follow the arithmetical order, is positive; we shall call this the leading element; every other element may be obtained from it by suitably interchanging the suffixes. The sign $+$ or $-$ is to be prefixed to any element according as the number of *inversions* of order in the line of suffixes is even or odd; for instance in the element $a_3b_2c_1$, 2 and 1 are out of their natural order, or inverted with respect to 3; 1 is inverted with respect to 2; hence there are three *inversions* and the sign of the element is negative; in the element $a_3b_1c_2$ there are two inversions, hence the sign is positive.

557. The determinant whose leading element is $a_1b_2c_3d_4\cdots$ may thus be expressed by the notation

$$*\Sigma \pm a_1b_2c_3d_4\cdots,$$

the $\Sigma\pm$ placed before the leading element indicating the aggregate of all the elements which can be obtained from it by suitable interchanges of suffixes and adjustment of signs.

Sometimes the determinant is still more simply expressed by enclosing the leading element within brackets; thus $(a_1b_2c_3d_4\cdots)$ is used as an abbreviation of $\Sigma \pm a_1b_2c_3d_4\cdots$.

* The Greek letter Sigma.

Ex. In the determinant $(a_1b_2c_3d_4e_5)$ what sign is to be prefixed to the element $a_4b_3c_1d_5e_2$?

Here 3, 1, and 2 are inverted with respect to 4; 1 and 2 are inverted with respect to 3, and 2 is inverted with respect to 5; hence there are six inversions and the sign of the element is positive.

558. Determinant of Lower Order. If in Art. 554, each of the constituents $b_1, c_1, \cdots k_1$ is equal to zero, the determinant reduces to a_1A_1; in other words it is equal to the product of a_1 and a determinant of the $(n-1)$th order, and we easily infer the following general theorem.

If each of the constituents of the first row or column of a determinant is zero except the first, and if this constituent is equal to m, the determinant is equal to m times that determinant of lower order which is obtained by omitting the first column and first row.

Also since by suitable interchange of rows and columns any constituent can be brought into the first place, it follows that if *any* row or column has all its constituents except one equal to zero, the determinant can be immediately expressed as a determinant of lower order.

This is sometimes useful in the reduction and simplification of determinants.

Ex. Find the value of

$$\begin{vmatrix} 30 & 11 & 20 & 38 \\ 6 & 3 & 0 & 9 \\ 11 & -2 & 36 & 3 \\ 19 & 6 & 17 & 22 \end{vmatrix}.$$

Diminish each constituent of the first column by twice the corresponding constituent in the second column, and each constituent of the fourth column by three times the corresponding constituent in the second column, and we obtain

$$\begin{vmatrix} 8 & 11 & 20 & 5 \\ 0 & 3 & 0 & 0 \\ 15 & -2 & 36 & 9 \\ 7 & 6 & 17 & 4 \end{vmatrix},$$

and since the second row has three zero constituents, this determinant

$$= 3\begin{vmatrix} 8 & 20 & 5 \\ 15 & 36 & 9 \\ 7 & 17 & 4 \end{vmatrix} = 3\begin{vmatrix} 8 & 20 & 5 \\ 8 & 19 & 5 \\ 7 & 17 & 4 \end{vmatrix} = 3\begin{vmatrix} 0 & 1 & 0 \\ 8 & 19 & 5 \\ 7 & 17 & 4 \end{vmatrix} = -3\begin{vmatrix} 8 & 5 \\ 7 & 4 \end{vmatrix} = 9.$$

EXAMPLES XLVII. b.

Calculate the values of the determinants:

1. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \end{vmatrix}.$

2. $\begin{vmatrix} 7 & 13 & 10 & 6 \\ 5 & 9 & 7 & 4 \\ 8 & 12 & 11 & 7 \\ 4 & 10 & 6 & 3 \end{vmatrix}.$

3. $\begin{vmatrix} a & 1 & 1 & 1 \\ 1 & a & 1 & 1 \\ 1 & 1 & a & 1 \\ 1 & 1 & 1 & a \end{vmatrix}.$

4. $\begin{vmatrix} 0 & 1 & 1 & 1 \\ 1 & b+c & a & a \\ 1 & b & c+a & b \\ 1 & c & c & a+b \end{vmatrix}.$

5. $\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}.$

6. $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix}.$

7. $\begin{vmatrix} 0 & x & y & z \\ x & 0 & z & y \\ y & z & 0 & x \\ z & y & x & 0 \end{vmatrix}.$

8. $\begin{vmatrix} 0 & x & y & z \\ -x & 0 & c & b \\ -y & -c & 0 & a \\ -z & -b & -a & 0 \end{vmatrix}.$

Solve the equations:

9. $4x - 5y + 2z = 11,$
$2x + 3y - z = 20,$
$7x - 4y + 3z = 33.$

10. $\dfrac{x}{2} + \dfrac{y}{3} + \dfrac{z}{4} = 7,$
$x + 2y + 3z = 48,$
$\dfrac{x}{3} - \dfrac{2y}{3} + \dfrac{z}{3} = 4.$

11. $2x - y + 3z - 2u = 14,$
$x + 7y + z - u = 13,$
$3x + 5y - 5z + 3u = 11,$
$4x - 3y + 2z - u = 21.$

12. $x + y + z = 1,$
$ax + by + cz = k,$
$a^2x + b^2y + c^2z = k^2.$

13. $ax + by + cz = k,$
$a^2x + b^2y + c^2z = k^2,$
$a^3x + b^3y + c^3z = k^3.$

14. $x + y + z + u = 1,$
$ax + by + cz + du = k,$
$a^2x + b^2y + c^2z + d^2u = k^2,$
$a^3x + b^3y + c^3z + d^3u = k^3.$

CHAPTER XLVIII.

THEORY OF EQUATIONS.

559. General Form of an Equation of the *n*th Degree. Let $p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$ be a rational integral function of x of n dimensions, and let us denote it by $f(x)$; then $f(x) = 0$ is the general type of a *rational integral equation* of the nth degree. Dividing throughout by p_0, we see that without any loss of generality we may take

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$$

as the *general form* of a rational integral equation of any degree.

Unless otherwise stated the coefficients $p_1, p_2, \cdots p_n$ will always be supposed rational.

If any of the coefficients $p_1, p_2, p_3, \cdots p_n$ are zero, the equation is said to be *incomplete*, otherwise it is called *complete*.

560. Any value of x which makes $f(x)$ vanish is called a **root** of the equation $f(x) = 0$.

561. We shall assume that every equation of the form $f(x) = 0$ has a root, real or imaginary. The proof of this proposition will be found in treatises on the *Theory of Equations;* it is beyond the range of the present work.

562. Divisibility of Equations. *If a is a root of the equation $f(x) = 0$, then is $f(x)$ exactly divisible by $x - a$.*

Divide the first member by $x - a$ until the remainder no longer contains x. Denote the quotient by Q, and the remainder, if there be one, by R. Then we have

$$f(x) = Q(x - a) + R = 0.$$

Now since a is a root of the equation $x = a$, therefore

$$Q(a - a) + R = 0,$$

hence

$$R = 0;$$

that is, the first member of the given equation is exactly divisible by $x - a$.

563. Conversely, *if the first member of* $f(x) = 0$ *is exactly divisible by* $x - a$, *then* a *is a root of the equation.*

For, the division being exact,

$$Q(x - a) = 0,$$

and the substitution of a for x satisfies the equation; hence a is a root.

DIVISION BY DETACHED COEFFICIENTS.

564. The work of dividing one multinomial by another may be abridged by writing only the coefficients of the terms. The following is an illustration.

Ex. Divide $3x^5 - 8x^4 - 5x^3 + 26x^2 - 33x + 26$ by $x^3 - 2x^2 - 4x + 8$.

$$\begin{array}{rrrrrrrl}
1+2+4-8) & 3 & -8 & -5 & +26 & -33 & +26 & (3-2+3\\
 & 3 & +6 & +12 & -24 & & & \\ \hline
 & & -2 & +7 & +2 & -33 & & \\
 & & -2 & -4 & -8 & +16 & & \\ \hline
 & & & 3 & -6 & -17 & +26 & \\
 & & & 3 & +6 & +12 & -24 & \\ \hline
 & & & & & -5 & +2 &
\end{array}$$

Thus the quotient is $3x^2 - 2x + 3$ and the remainder is $-5x + 2$.

It should be noticed that in writing the divisor, the sign of every term *except the first* has been changed; this enables us *to replace the process of subtraction by that of addition* at each successive stage of the work.

HORNER'S METHOD OF SYNTHETIC DIVISION.

565. For convenience we again give an explanation of **Horner's Method of Synthetic Division,** which has already been considered in Art. 63.

Let us take the example of the preceding article. The arrangement of the work is as follows:

$$\begin{array}{r|rrrrrr}
1 & 3 & -8 & -5 & +26 & -33 & +26 \\
2 & & 6 & +12 & -24 & & \\
4 & & & -4 & -8 & +16 & \\
-8 & & & & 6 & +12 & -24 \\
\hline
 & 3 & -2 & +3 & +0 & -5 & +2
\end{array}$$

Explanation. The column of figures to the left of the vertical line consists of the coefficients of the divisor, the sign of each after the first being changed; the second horizontal line is obtained by multiplying 2, 4, -8 by 3, the first term of the quotient. We then add the terms in the second column to the right of the vertical line; this gives -2, which is the coefficient of the second term of the quotient. With the coefficient thus obtained we form the next horizontal line, and add the terms in the third column; this gives 3, which is the coefficient of the third term of the quotient.

By adding up the other columns we get the coefficients of the terms in the remainder.

566. In employing this method in the following articles our divisor will be of the form $x \pm a$, which enables us to still further simplify the work, as the following example shows:

Ex. Find the quotient and remainder when $3x^7 - x^6 + 31x^4 + 21x + 5$ is divided by $x + 2$.

$$\begin{array}{rrrrrrrr|l}
3 & -1 & 0 & 31 & 0 & 0 & 21 & 5 & -2 \\
 & -6 & 14 & -28 & -6 & 12 & -24 & 6 & \\
\hline
3 & -7 & 14 & 3 & -6 & 12 & -3 & 11 &
\end{array}$$

Thus the quotient is $3x^6 - 7x^5 + 14x^4 + 3x^3 - 6x^2 + 12x - 3$, and the remainder is 11.

Explanation. The first horizontal line contains the coefficients of the dividend, *zero coefficients being used to represent terms corresponding to powers of x which are absent.* The divisor is written at the right of this line *with its sign changed* (Art. 564) and 1, the coefficient of x, omitted. The first term of the third horizontal line, which contains the quotient, is the result of dividing 3, the coefficient of x^7 in the dividend, by 1, the coefficient of x in the divisor. This is then *multiplied* by the divisor -2, and the result is -6, the first term of the second horizontal line; the sum of -1 and -6 gives -7, the second term of the quotient, which multiplied by -2 gives 14 for the second term of the second horizontal line; the addition of 14 and 0 gives 14 for the third term of the quotient, which multiplied by -2 gives -28 for the third term of the second line, and so on.

567. Number of Roots. *Every equation of the* nth *degree has* n *roots, and no more.*

Denote the given equation by $f(x)=0$, where

$$f(x)=x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_n.$$

The equation $f(x)=0$ has a root, real or imaginary; let this be denoted by a_1; then $f(x)$ is divisible by $x-a_1$, so that

$$f(x)=(x-a_1)f_1(x),$$

where $f_1(x)$ is a rational integral function of $n-1$ dimensions. Again, the equation $f_1(x)=0$ has a root, real or imaginary; let this be denoted by a_2; then $f_1(x)$ is divisible by $x-a_2$, so that

$$f_1(x)=(x-a_2)f_2(x),$$

where $f_2(x)$ is a rational integral function of $n-2$ dimensions.

Thus $$f(x)=(x-a_1)(x-a_2)f_2(x).$$

Proceeding in this way, we obtain

$$f(x)=(x-a_1)(x-a_2)\cdots(x-a_n).$$

Hence the equation $f(x)=0$ has n roots, since $f(x)$ vanishes when x has any of the values $a_1, a_2, a_3, \cdots a_n$.

Also the equation cannot have more than n roots; for if x has any value different from any of the quantities $a_1, a_2, a_3, \cdots a_n$, all the factors on the right are different from zero, and therefore $f(x)$ cannot vanish for that value of x.

In the above investigation some of the quantities $a_1, a_2, a_3, \cdots a_n$ may be equal; in this case, however, we shall suppose that the equation has still n roots, although these are not all different.

568. Depression of Equations. If one root of an equation is known it is evident from the preceding paragraph that we may by division reduce or *depress* the equation to one of the next lower degree containing the remaining roots. So if k

roots are known we may depress the equation to one of the $(n-k)$th degree. All the roots but two being known, the depressed equation is a quadratic from which the remaining roots are readily obtained.

569. Formation of Equations. Since $f(x)=(x-a_1)(x-a_2)\cdots(x-a_n)$ [Art. 567], we see that an equation may be formed by *subtracting each root from the unknown quantity and placing the continued product of the binomial factors thus formed equal to 0.*

Ex. Form the equation whose roots are 1, -2, and $\frac{1}{2}$.

$$(x-1)(x+2)(x-\tfrac{1}{2})=0;$$
$$\therefore\ 2x^3+x^2-5x+2=0.$$

EXAMPLES XLVIII. a.

1. Show that 4 is a root of $x^3-5x^2-2x+24=0$.
2. Show that $+3$ is a root of $x^3+7x^2+7x+15=0$.
3. Show that $-\frac{1}{3}$ is a root of $6x^3+17x^2-4x-3=0$.
4. Show that $\frac{4}{5}$ is a root of $10x^3-3x^2-9x+4=0$.
5. One root of $x^3+6x^2-6x-63=0$ is 3; find the others.
6. One root of $x^3-23x^2+166x-378=0$ is 7; find the others.
7. One root of $x^3-2x^2+6x-9\frac{2}{27}=0$ is $\frac{5}{3}$; what are the others?
8. Two roots of $x^4-15x^2+10x+24=0$ are 2 and 3; find the others.
9. Two roots of $x^4-3x^3-21x^2+43x+60=0$ are 3 and 5; find the others.
10. One root of $x^3+2ax^2+5a^2x+4a^3=0$ is $-a$; what are the others?
11. Form the equation whose roots are -1, -2, and -5.
12. Form the equation whose roots are -2, -3, $+\frac{1}{2}$, and $-\frac{1}{3}$.

570. Relations between the Roots and the Coefficients. Let us denote the equation by

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n=0,$$

and the roots by $a, b, c, \cdots k$; then we have identically

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p^n$$
$$=(x-a)(x-b)(x-c)\cdots(x-k);$$

hence, by multiplication, we have

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$$
$$= x^n - S_1x^{n-1} + S_2x^{n-2} - \cdots + (-1)^{n-1}S_{n-1}x + (-1)^nS_n.$$

Equating the coefficients of like powers of x in this identity, we have

$$-p_1 = S_1,$$
$$p_2 = S_2,$$
$$-p_3 = S_3,$$
$$(-1)^np_n = S_n,$$

in which S_1 stands for the sum of the roots $a, b, c \cdots k$; S_2 stands for the sum of the products of the roots taken two at a time, and so on to S_n, which equals the continued product of all the roots. That is:

(1) The coefficient of the second term *with its sign changed* equals the sum of the roots.

(2) The coefficient of the third term equals the sum of all the products of the roots taken two at a time.

(3) The coefficient of the fourth term *with its sign changed* equals the sum of all the products of the roots taken three at a time, and so on.

(4) The last term equals the continued product of all the roots, the sign being + or − according as n is even or odd.

571. It follows that if the equation is in the general form:

(1) The sum of the roots is zero if the second term is wanting.

(2) One root, at least, is zero if the last term is wanting.

572. The student might suppose that the relations established in the preceding article would enable him to solve any proposed equation; for the number of the relations is equal to the number of the roots. A little reflection will show that this is not the case; for suppose we eliminate any $n-1$ of the quantities $a, b, c, \cdots k$, and so obtain an equation to determine the remaining one; then since these quantities are involved symmetrically in each of the equations, it is clear that we shall always obtain an equation having the same coefficients: this equation is therefore the original equation with some one of the roots $a, b, c, \cdots k$ substituted for x.

Let us take for example the equation

$$x^3 + p_1x^2 + p_2x + p_3 = 0;$$

and let a, b, c be the roots; then

$$a + b + c = -p_1,$$
$$ab + ac + bc = +p_2,$$
$$abc = -p_3.$$

Multiply these equations by $a^2, -a, 1$ respectively and add; thus $\qquad a^3 = -p_1a^2 - p_2a - p_3,$

that is, $\qquad a^3 + p_1a^2 + p_2a + p_3 = 0,$

which is the original equation with a in the place of x.

The above process of elimination is quite general, and is applicable to equations of any degree.

573. If two or more of the roots of an equation are connected by an assigned relation, the properties proved in Art. 570 will sometimes enable us to obtain the complete solution.

Ex. 1. Solve the equation $4x^3 - 24x^2 + 23x + 18 = 0$, having given that the roots are in arithmetical progression.

Denote the roots by $a - b, a, a + b$; then the sum of the roots is $3a$; the sum of the products of the roots two at a time is $3a^2 - b^2$; and the product of the roots is $a(a^2 - b^2)$; hence we have the equations

$$3a = 6, \quad 3a^2 - b^2 = \tfrac{23}{4}, \quad a(a^2 - b^2) = -\tfrac{9}{2};$$

from the first equation we find $a = 2$, and from the second $b = \pm\frac{5}{2}$, and since these values satisfy the third, the three equations are consistent. Thus the roots are $-\frac{1}{2}, 2, \frac{9}{2}$.

Ex. 2. Solve the equation $24x^3 - 14x^2 - 63x + 45 = 0$, one root being double another.

Denote the roots by $a, 2a, b$; then we have

$$3a + b = \tfrac{7}{12}, \quad 2a^2 + 3ab = -\tfrac{21}{8}, \quad 2a^2b = -\tfrac{15}{8}.$$

From the first two equations, we obtain

$$8a^2 - 2a - 3 = 0;$$

$$\therefore a = \tfrac{3}{4} \text{ or } -\tfrac{1}{2}, \text{ and } b = -\tfrac{5}{3} \text{ or } \tfrac{25}{12}.$$

It will be found on trial that the values $a = -\frac{1}{2}$, $b = \frac{25}{12}$ do not satisfy the third equation $2a^2b = -\frac{15}{8}$; hence we are restricted to the values $a = \frac{3}{4}$, $b = -\frac{5}{3}$.

Thus the roots are $\frac{3}{4}, \frac{3}{2}, -\frac{5}{3}$.

574. Although we may not be able to find the roots of an equation, we can make use of the relations proved in Art. 570 to determine the values of symmetrical * functions of the roots.

Ex. Find the sum of the squares and of the cubes of the roots of the equation $x^3 - px^2 + qx - r = 0$.

Denote the roots by a, b, c; then $a + b + c = p$, $bc + ca + ab = q$.

Now $a^2 + b^2 + c^2 = (a + b + c)^2 - 2(bc + ca + ab) = p^2 - 2q$.

Again, substitute a, b, c for x in the given equation and add; thus

$$a^3 + b^3 + c^3 - p(a^2 + b^2 + c^2) + q(a + b + c) - 3r = 0;$$

$$\therefore\ a^3 + b^3 + c^3 = p(p^2 - 2q) - pq + 3r = p^3 - 3pq + 3r.$$

EXAMPLES XLVIII. b.

Form the equation whose roots are:

1. $\frac{2}{3}, \frac{3}{2}, \pm\sqrt{3}$. **2.** $0, 0, 2, 2, -3, -3$.

3. $2, 2, -2, -2, 0, 5$. **4.** $a + b,\ a - b,\ -a + b,\ -a - b$.

Solve the equations:

5. $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$, two roots being 1 and 7.

6. $4x^3 + 16x^2 - 9x - 36 = 0$, the sum of two of the roots being zero.

7. $4x^3 + 20x^2 - 23x + 6 = 0$, two of the roots being equal.

8. $3x^3 - 26x^2 + 52x - 24 = 0$, the roots being in geometrical progression.

9. $2x^3 - x^2 - 22x - 24 = 0$, two of the roots being in the ratio of $3:4$.

10. $24x^3 + 46x^2 + 9x - 9 = 0$, one root being double another of the roots.

11. $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$, two of the roots being equal but opposite in sign.

* A function is said to be *symmetrical* with respect to its variables when its value is unaltered by the interchange of any pair of them: thus $x + y + z$, $bc + ca + ab$, $x^3 + y^3 + z^3 - xyz$ are symmetrical functions of the first, second, and third degrees respectively. [See Art. 319.]

12. $54x^3 - 39x^2 - 26x + 16 = 0$, the roots being in geometrical progression.

13. $32x^3 - 48x^2 + 22x - 3 = 0$, the roots being in arithmetical progression.

14. $6x^4 - 29x^3 + 40x^2 - 7x - 12 = 0$, the product of two of the roots being 2.

15. $x^4 - 2x^3 - 21x^2 + 22x + 40 = 0$, the roots being in arithmetical progression.

16. $27x^4 - 195x^3 + 494x^2 - 520x + 192 = 0$, the roots being in geometrical progression.

17. $18x^3 + 81x^2 + 121x + 60 = 0$, one root being half the sum of the other two.

18. Find the sum of the squares and of the cubes of the roots of $x^4 + qx^2 + rx + s = 0$.

575. Fractional Roots. *An equation whose coefficients are integers, that of the first term being unity, cannot have a rational fraction as a root.*

If possible suppose the equation

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-2}x^2 + p_{n-1}x + p_n = 0$$

has for a root a rational fraction in its lowest terms, represented by $\frac{a}{b}$. Substituting this value for x and multiplying through by b^{n-1}, we have

$$\frac{a^n}{b} + p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_{n-1}ab^{n-2} + p_nb^{n-1} = 0.$$

Transposing,

$$-\frac{a^n}{b} = p_1a^{n-1} + p_2a^{n-2}b + \cdots + p_{n-1}ab^{n-2} + p_nb^{n-1}.$$

This result is impossible, since it makes a fraction in its lowest terms equal to an integer. Hence a rational fraction cannot be a root of the given equation.

576. Imaginary Roots. *In an equation with real coefficients imaginary roots occur in pairs.*

Suppose that $f(x) = 0$ is an equation with real coefficients, and suppose that it has an imaginary root $a + ib$; we shall show that $a - ib$ is also a root.

The factor of $f(x)$ corresponding to these two roots is

$$(x-a-ib)(x-a+ib), \text{ or } (x-a)^2+b^2.$$

Let $f(x)$ be divided by $(x-a)^2+b^2$; denote the quotient by Q, and the remainder, if any, by $Rx+R'$; then

$$f(x)=Q\{(x-a)^2+b^2\}+Rx+R'.$$

In this identity put $x=a+ib$, then $f(x)=0$ by hypothesis; also $(x-a)^2+b^2=0$; hence $R(a+ib)+R'=0$.

Equating to zero the real and imaginary parts,

$$Ra+R'=0, \quad Rb=0;$$

and b by hypothesis is not zero,

$$\therefore\ R=0 \text{ and } R'=0.$$

Hence $f(x)$ is exactly divisible by $(x-a)^2+b^2$, that is, by

$$(x-a-ib)(x-a+ib);$$

hence $x=a-ib$ is also a root.

577. In the preceding article we have seen that if the equation $f(x)=0$ has a pair of imaginary roots $a\pm ib$, then $(x-a)^2+b^2$ is a factor of the expression $f(x)$.

Suppose that $a\pm ib$, $c\pm id$, $e\pm ig$, $\cdots$ are the imaginary roots of the equation $f(x)=0$, and that * $\phi(x)$ is the product of the quadratic factors corresponding to these imaginary roots; then

$$\phi(x)=\{(x-a)^2+b^2\}\{(x-c)^2+d^2\}\{(x-e)^2+g^2\}\cdots.$$

Now each of these factors is positive for every real value of x; hence $\phi(x)$ is always positive for real values of x.

578. As in Art. 576 we may show that in an equation with *rational* coefficients, surd roots enter in pairs; that is, if $a+\sqrt{b}$ is a root then $a-\sqrt{b}$ is also a root.

Ex. 1. Solve the equation $6x^4-13x^3-35x^2-x+3=0$, having given that one root is $2-\sqrt{3}$.

* The Greek letter Phi.

Since $2-\sqrt{3}$ is a root, we know that $2+\sqrt{3}$ is also a root, and corresponding to this pair of roots we have the quadratic factor x^2-4x+1.

Also

$$6x^4-13x^3-35x^2-x+3=(x^2-4x+1)(6x^2+11x+3);$$

hence the other roots are obtained from

$$6x^2+11x+3=0, \text{ or } (3x+1)(2x+3)=0;$$

thus the roots are $\quad -\frac{1}{3},\ -\frac{3}{2},\ 2+\sqrt{3},\ 2-\sqrt{3}$.

Ex. 2. Form the equation of the fourth degree with rational coefficients, one of whose roots is $\sqrt{2}+\sqrt{-3}$.

Here we must have $\sqrt{2}+\sqrt{-3}$, $\sqrt{2}-\sqrt{-3}$ as one pair of roots, and $-\sqrt{2}+\sqrt{-3}$, $-\sqrt{2}-\sqrt{-3}$ as another pair.

Corresponding to the first pair we have the quadratic factor $x^2-2\sqrt{2}x+5$, and corresponding to the second pair we have the quadratic factor

$$x^2+2\sqrt{2}x+5.$$

Thus the required equation is

$$(x^2+2\sqrt{2}x+5)(x^2-2\sqrt{2}x+5)=0,$$

or $$(x^2+5)^2-8x^2=0,$$

or $$x^4+2x^2+25=0.$$

EXAMPLES XLVIII. c.

Solve the equations:

1. $3x^4-10x^3+4x^2-x-6=0$, one root being $\dfrac{1+\sqrt{-3}}{2}$.
2. $x^4-36x^2+72x-36=0$, one root being $3-\sqrt{3}$.
3. $x^4+4x^3+5x^2+2x-2=0$, one root being $-1+\sqrt{-1}$.
4. $x^4+4x^3+6x^2+4x+5=0$, one root being $\sqrt{-1}$.

TRANSFORMATION OF EQUATIONS.

579. The discussion of an equation is sometimes simplified by transforming it into another equation whose roots bear some assigned relation to those of the one proposed. Such transformations are especially useful in the solution of cubic equations.

580. To transform an equation into another whose roots are those of the original equation with their signs changed.

Let $f(x)=0$ be the equation.

Put $-y$ for x; then the equation $f(-y)=0$ is satisfied by every root of $f(x)=0$ with its sign changed; thus the required equation is $f(-y)=0$.

If the given equation is

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n=0,$$

then it is evident that the required equation will be

$$y^n-p_1y^{n-1}+p_2y^{n-2}-\cdots+(-1)^{n-1}p_{n-1}y+(-1)^np_n=0;$$

therefore the transformed equation is obtained from the original equation by *changing the sign of every alternate term beginning with the second.*

NOTE. If any term of the given equation is missing it must be supplied with zero as a coefficient.

Ex. Transform the equation $x^4-17x^2-20x-6=0$ into another which shall have the same roots numerically with contrary signs. We may write the equation thus:

$$x^4+0x^3-17x^2-20x-6=0.$$

By the rule, we have

$$x^4-0x^3-17x^2+20x-6=0,$$

or

$$x^4-17x^2+20x-6=0.$$

581. To transform an equation into another whose roots are equal to those of the original equation multiplied by a given factor.

Let $f(x)=0$ be the equation, and let q denote the given quantity. Put $y=qx$, so that when x has any particular value, y is q times as large; then $x=\dfrac{y}{q}$, and the required equation is

$$\left(\frac{y}{q}\right)^n+p_1\left(\frac{y}{q}\right)^{n-1}+p_2\left(\frac{y}{q}\right)^{n-2}+\cdots+p_{n-1}\left(\frac{y}{q}\right)+p_n=0.$$

Multiplying by q^n, we have

$$y^n+p_1qy^{n-1}+p_2q^2y^{n-2}+\cdots+p_{n-1}q^{n-1}y+p_nq^n=0.$$

Therefore the transformed equation is obtained from the original equation by *multiplying the second term by the given factor, the third term by the square of this factor, and so on.*

582. The chief use of this transformation is *to clear an equation of fractional coefficients.*

Ex. Remove fractional coefficients from the equation

$$2x^3 - \tfrac{3}{2}x^2 - \tfrac{1}{8}x + \tfrac{3}{16} = 0.$$

Put $x = \dfrac{y}{q}$ and multiply each term by q^3; thus

$$2y^3 - \tfrac{3}{2}qy^2 - \tfrac{1}{8}q^2y + \tfrac{3}{16}q^3 = 0.$$

By putting $q = 4$ all the terms become integral, and on dividing by 2, we obtain

$$y^3 - 3y^2 - y + 6 = 0.$$

583. To transform an equation into another whose roots exceed those of the original equation by a given quantity.

Let $f(x) = 0$ be the equation, and let h be the given quantity. Assume $y = x + h$, so that for any particular value of x, the value of y is greater by h; thus $x = y - h$, and the required equation is $f(y - h) = 0$.

Similarly if the roots are to be *less* by h, we assume $y = x - h$, from which we obtain $x = y + h$, and the required equation is $f(y + h) = 0$.

584. If n is small, this method of transformation is effected with but little trouble. For equations of a higher degree the following method is to be preferred:

Let $\quad f(x) = p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n;$

put $x = y + h$, and suppose that $f(x)$ then becomes

$$q_0y^n + q_1y^{n-1} + q_2y^{n-2} + \cdots + q_{n-1}y + q_n.$$

Now $y = x - h$; hence we have the identity

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n$$
$$= q_0(x-h)^n + q_1(x-h)^{n-1} + \cdots + q_{n-1}(x-h) + q_n;$$

therefore q_n is the remainder found by dividing $f(x)$ by $x - h$; also the quotient arising from the division is

$$q_0(x-h)^{n-1} + q_1(x-h)^{n-2} + \cdots + q_{n-1}.$$

Similarly q_{n-1} is the remainder found by dividing the last expression by $x-h$, and the quotient arising from the division is

$$q_0(x-h)^{n-2}+q_1(x-h)^{n-3}+\cdots+q_{n-2};$$

and so on. Thus $q_n, q_{n-1}, q_{n-2}, \cdots$ may be readily found by Synthetic Division. The last quotient is q_0, and is obviously equal to p_0.

Hence to obtain the transformed equation,

Divide $f(x)$ by $x \pm h$ according as the roots are to be greater or less by h than those of the original equation, and the remainder will be the last term of the required equation. Divide the quotient thus found by $x \pm h$, and the remainder will be the coefficient of the last term but one of the required equation; and so on.

Ex. Find the equation whose roots exceed by 2 the roots of the equation

$$4x^4+32x^3+83x^2+76x+21=0.$$

The required equation will be obtained by substituting $x-2$ for x in the proposed equation; hence in Horner's process we employ $x+2$ as divisor, and the calculation is performed as follows:

4	32	83	76	21	$\underline{\vert +2}$
4	24	35	6	$\vert$ 9	
4	16	3	$\vert$ 0		
4	8	$\vert -13$			
4	$\vert$ 0				
4					

Thus the transformed equation is

$$4x^4-13x^2+9=0, \text{ or } (4x^2-9)(x^2-1)=0.$$

The roots of this equation are $+\frac{3}{2}, -\frac{3}{2}, +1, -1$; hence the roots of the given equation are

$$-\tfrac{1}{2}, \; -\tfrac{7}{2}, \; -1, \; -3.$$

585. To transform a complete equation into another which wants an assigned term.

The chief use of the substitution in the preceding article is to remove some assigned term from an equation.

Let the given equation be

$$p_0x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0;$$

then if $y = x - h$, we obtain the new equation

$$p_0(y+h)^n + p_1(y+h)^{n-1} + p_2(y+h)^{n-2} + \cdots + p_n = 0,$$

which, when arranged in descending powers of y, becomes

$$p_0y^n + (np_0h + p_1)\,y^{n-1} + \left\{\frac{n(n-1)}{\underline{|2}}p_0h^2 + (n-1)p_1h + p_2\right\}y^{n-2} + \cdots = 0.$$

If the term to be removed is the second, we put $np_0h + p_1 = 0$, so that $h = -\frac{p_1}{np_0}$; if the term to be removed is the third, we put

$$\frac{n(n-1)}{\underline{|2}}p_0h^2 + (n-1)p_1h + p_2 = 0,$$

and so obtain a quadratic to find h; and, similarly, we may remove any other assigned term.

Sometimes it will be more convenient to proceed as in the following example.

Ex. Remove the second term from the equation

$$px^3 + qx^2 + rx + s = 0.$$

Let a, b, c be the roots, so that $a + b + c = -\frac{q}{p}$. Then if we increase each of the roots by $\frac{q}{3p}$, in the transformed equation the sum of the roots will be equal to $-\frac{q}{p} + \frac{q}{p}$; that is, the coefficient of the second term will be zero.

Hence the required transformation will be effected by substituting $x - \frac{q}{3p}$ for x in the given equation.

As the general type of a cubic equation can be reduced to a more simple form by removing the second term, the student should carefully notice that the transformation is effected by substituting x minus the coefficient of the second term divided by the degree of the equation, for x in the given equation.

586. To transform an equation into another whose roots are the reciprocals of the roots of the proposed equation.

Let $f(x)=0$ be the proposed equation; put $y=\frac{1}{x}$, so that $x=\frac{1}{y}$; then the required equation is $f\left(\frac{1}{y}\right)=0$.

One of the chief uses of this transformation is to obtain the values of expressions which involve symmetrical functions of negative powers of the roots.

Ex. If a, b, c are the roots of the equation

$$x^3-px^2+qx-r=0,$$

find the value of

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}.$$

Write $\frac{1}{y}$ for x, multiply by y^3, and change all the signs; then the resulting equation

$$ry^3-qy^2+py-1=0,$$

has for its roots

$$\frac{1}{a},\ \frac{1}{b},\ \frac{1}{c};$$

hence

$${}^*\Sigma\frac{1}{a}=\frac{q}{r},\quad \Sigma\frac{1}{ab}=\frac{p}{r};$$

$$\therefore\ \Sigma\frac{1}{a^2}=\frac{q^2-2pr}{r^2}.$$

587. Reciprocal Equations. If an equation is unaltered by changing x into $\frac{1}{x}$, it is called a *reciprocal equation.*

If the given equation is

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-2}x^2+p_{n-1}x+p_n=0,$$

the equation obtained by writing $\frac{1}{x}$ for x, and clearing of fractions, is

$$p_nx^n+p_{n-1}x^{n-1}+p_{n-2}x^{n-2}+\cdots+p_2x^2+p_1x+1=0.$$

If these two equations are the same, we must have

$$p_1=\frac{p_{n-1}}{p_n},\ p_2=\frac{p_{n-2}}{p_n},\ \cdots,\ p_{n-2}=\frac{p_2}{p_n},\ p_{n-1}=\frac{p_1}{p_n},\ p_n=\frac{1}{p_n};$$

* $\Sigma\frac{1}{a}$ stands for the sum of all the terms of which $\frac{1}{a}$ is the type.

from the last result we have $p_n = \pm 1$, and thus we have two classes of reciprocal equations.

(i.) If $p_n = 1$, then

$$p_1 = p_{n-1},\ p_2 = p_{n-2},\ p_3 = p_{n-3}, \cdots;$$

that is, the coefficients of terms equidistant from the beginning and end are equal.

(ii.) If $p_n = -1$, then

$$p_1 = -p_{n-1},\ p_2 = -p_{n-2},\ p_3 = -p_{n-3}, \cdots;$$

hence if the equation is of $2m$ dimensions $p_m = -p_m$, or $p_m = 0$. In this case the coefficients of terms equidistant from the beginning and end are equal in magnitude and opposite in sign, and if the equation is of an even degree the middle term is wanting.

588. Standard Form of Reciprocal Equations. Suppose that $f(x) = 0$ is a reciprocal equation.

If $f(x) = 0$ is of the first class and of an odd degree it has a root -1; so that $f(x)$ is divisible by $x + 1$. If $\phi(x)$ is the quotient, then $\phi(x) = 0$ is a reciprocal equation of the first class and of an even degree.

If $f(x) = 0$ is of the second class and of an odd degree, it has a root $+1$; in this case $f(x)$ is divisible by $x - 1$, and as before $\phi(x) = 0$ is the reciprocal equation of the first class and of an even degree.

If $f(x) = 0$ is of the second class and of an even degree, it has a root $+1$ and a root -1; in this case $f(x)$ is divisible by $x^2 - 1$, and as before $\phi(x) = 0$ is a reciprocal equation of the first class of an even degree.

Hence *any reciprocal equation is of an even degree with its last term positive, or can be reduced to this form;* which may therefore be considered as the standard form of reciprocal equations.

589. A reciprocal equation of the standard form can be reduced to an equation of half its dimensions.

Let the equation be

$$ax^{2m} + bx^{2m-1} + cx^{2m-2} + \cdots + kx^m + \cdots + cx^2 + bx + a = 0;$$

dividing by x^m and rearranging the terms, we have

$$a\left(x^m + \frac{1}{x^m}\right) + b\left(x^{m-1} + \frac{1}{x^{m-1}}\right) + c\left(x^{m-2} + \frac{1}{x^{m-2}}\right) + \cdots + k = 0.$$

Now $$x^{p+1} + \frac{1}{x^{p+1}} = \left(x^p + \frac{1}{x^p}\right)\left(x + \frac{1}{x}\right) - \left(x^{p-1} + \frac{1}{x^{p-1}}\right);$$

hence writing z for $x + \frac{1}{x}$, and giving to p in succession the values 1, 2, 3 $\cdots$ we obtain

$$x^2 + \frac{1}{x^2} = z^2 - 2;$$

$$x^3 + \frac{1}{x^3} = z(z^2 - 2) - z = z^3 - 3z;$$

$$x^4 + \frac{1}{x^4} = z(z^3 - 3z) - (z^2 - 2) = z^4 - 4z^2 + 2;$$

and so on; and generally $x^m + \frac{1}{x^m}$ is of m dimensions in z, and therefore the equation in z is of m dimensions.

590. To find the equation whose roots are the squares of those of a proposed equation.

Let $f(x) = 0$ be the given equation; by putting $y = x^2$, we have $x = \sqrt{y}$, and therefore the required equation is $f(\sqrt{y}) = 0$.

Ex. Find the equation whose roots are the squares of those of the equation

$$x^3 + p_1x^2 + p_2x + p_3 = 0.$$

Putting $x = \sqrt{y}$, and transposing, we have

$$(y + p_2)\sqrt{y} = -(p_1y + p_3);$$

whence $$(y^2 + 2p_2y + p_2^2)\,y = p_1^2y^2 + 2p_1p_3y + p_3^2,$$

or $$y^3 + (2p_2 - p_1^2)\,y^2 + (p_2^2 - 2p_1p_3)\,y - p_3^2 = 0.$$

EXAMPLES XLVIII. d.

1. Transform the equation $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$ into another which shall have the same roots with contrary signs.

2. Transform the equation $2x^3 - 4x^2 + 7x - 3 = 0$ into another whose roots shall be those of the first multiplied by 3.

3. Transform the equation $x^3 - 7x - 6 = 0$ into another whose roots shall be those of the first multiplied by $-\frac{2}{3}$.

4. Transform the equation $x^3 - 4x^2 + \frac{1}{4}x - \frac{1}{9} = 0$ into another with integral coefficients, and unity for the coefficient of the first term.

5. Transform the equation $3x^4 - 5x^3 + x^2 - x + 1 = 0$ into another the coefficient of whose first term is unity.

6. Transform the equation $x^4 + 10x^3 + 39x^2 + 76x + 65 = 0$ into another whose roots shall be greater by 4.

7. Transform the equation $x^4 - 12x^3 + 17x^2 - 9x + 7 = 0$ into another whose roots shall be less by 3.

8. Diminish by 1 the roots of the equation

$$2x^4 - 13x^2 + 10x - 19 = 0.$$

9. Find the equation whose roots are greater by 4 than the corresponding roots of $x^4 + 16x^3 + 72x^2 + 64x - 129 = 0$.

10. Solve the equation $3x^3 - 22x^2 + 48x - 32 = 0$, the roots of which are in harmonical progression.

11. The roots of $x^3 - 11x^2 + 36x - 36 = 0$ are in harmonical progression; find them.

Remove the second term from the equations:

12. $x^3 - 6x^2 + 10x - 3 = 0$.

13. $x^4 + 4x^3 + 2x^2 - 4x - 2 = 0$.

14. Transform the equation $x^3 - \dfrac{x}{4} - \dfrac{3}{4} = 0$ into one whose roots exceed by $\frac{3}{2}$ the corresponding roots of the given equation.

15. Diminish by 3 the roots of the equation

$$x^5 - 4x^4 + 3x^2 - 4x + 6 = 0.$$

16. Find the equation each of whose roots is greater by unity than a root of the equation $x^3 - 5x^2 + 6x - 3 = 0$.

17. Find the equation whose roots are the squares of the roots of

$$x^4 + x^3 + 2x^2 + x + 1 = 0.$$

18. Form the equation whose roots are the cubes of the roots of

$$x^3 + 3x^2 + 2 = 0.$$

If a, b, c are the roots of $x^3 + qx + r = 0$, form the equation whose roots are

19. ka^{-1}, kb^{-1}, kc^{-1}.

20. b^2c^2, c^2a^2, a^2b^2.

21. $\frac{b+c}{a^2}$, $\frac{c+a}{b^2}$, $\frac{a+b}{c^2}$.

22. $bc + \frac{1}{a}$, $ca + \frac{1}{b}$, $ab + \frac{1}{c}$.

Solve the equations:

23. $2x^4 + x^3 - 6x^2 + x + 2 = 0$.

24. $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.

25. $x^5 - 5x^4 + 9x^3 - 9x^2 + 5x - 1 = 0$.

26. $4x^6 - 24x^5 + 57x^4 - 73x^3 + 57x^2 - 24x + 4 = 0$.

DESCARTES' RULE OF SIGNS.

591. When each term of a series has one of the signs + and − before it, a *continuation* or *permanence* occurs when the signs of two successive terms are the *same:* and a *change* or *variation* occurs when the signs of two successive terms are *opposite.*

592. Descartes' Rule. *In any equation, the number of positive roots cannot exceed the number of variations of sign, and in any complete equation the number of negative roots cannot exceed the number of permanences of sign.*

Suppose that the signs of the terms in a multinomial are + + − − + − − − + − + −; we shall show that if this multinomial is multiplied by a binomial whose signs are + −, there will be at least one more change of sign in the product than in the original multinomial.

Writing only the signs of the terms in the multiplication, we have

$$\begin{array}{ccccccccccccc}
+ & + & - & - & + & - & - & - & + & - & + & - & \\
+ & - & & & & & & & & & & & \\
\hline
+ & + & - & - & + & - & - & - & + & - & + & - & \\
 & - & - & + & + & - & + & + & + & - & + & - & + \\
\hline
+ & \pm & - & \mp & + & - & \mp & \mp & + & - & + & - & +
\end{array}$$

a double sign, spoken of as an *ambiguity*, being placed wherever there is a doubt as to whether the sign of a term is positive or negative.

Examining the product we see that

(i.) An ambiguity replaces each continuation of sign in the original multinomial;

(ii.) The signs before and after an ambiguity or set of ambiguities are unlike;

(iii.) A change of sign is introduced at the end.

Let us take the most unfavorable case and suppose that all the ambiguities are replaced by continuations; from (ii.) we see that the number of changes of sign will be the same whether we take the upper or the lower signs; let us take the upper; thus the number of changes of sign cannot be less than in

$$+ + - - + - - - + - + - +,$$

and this series of signs is the same as in the original multinomial with an additional change of sign at the end.

If then we suppose the factors corresponding to the negative and imaginary roots to be already multiplied together, each factor $x-a$ corresponding to a positive root introduces at least one change of sign; therefore no equation can have more positive roots than it has changes of sign.

To prove the second part of Descartes' Rule, let us suppose the equation complete and substitute $-y$ for x; then the *permanences* of sign in the original equation become *variations* of sign in the transformed equation. Now the transformed equation cannot have more *positive* roots than it has *variations* of sign, hence the original equation cannot have more *negative* roots than it has *permanences* of sign.

Whether the equation $f(x)=0$ be *complete* or *incomplete* its roots are equal to those of $f(-x)$ but opposite to them in sign; therefore the negative roots of $f(x)=0$ are the positive roots of $f(-x)=0$; but the number of these positive roots cannot exceed the number of *variations* of sign in $f(-x)$; that is, the number of negative roots of $f(x)=0$ cannot exceed the number of *variations* of sign in $f(-x)$.

We may therefore enunciate *Descartes' Rule* as follows:

An equation $f(x)=0$ *cannot have more positive roots than there are variations of sign in* $f(x)$, *and cannot have more negative roots than there are variations of sign in* $f(-x)$.

Ex. Consider the equation $x^9+5x^8-x^3+7x+2=0$.

Here there are two changes of sign, therefore there are at most two positive roots.

Again $f(-x)=-x^9+5x^8+x^3-7x+2$, and here there are three changes of sign, therefore the given equation has at most three negative roots, and therefore it must have at least four imaginary roots.

593. It is very evident that the following results are included in the preceding article.

(i.) If the coefficients are all positive, the equation has no positive root; thus the equation $x^5+x^3+2x+1=0$ cannot have a positive root.

(ii.) If the coefficients of the even powers of x are all of one sign, and the coefficients of the odd powers are all of the contrary sign, the equation has no negative root; thus the equation

$$x^7+x^5-2x^4+x^3-3x^2+7x-5=0$$

cannot have a negative root.

EXAMPLES XLVIII. e.

Find the nature of the roots of the following equations:

1. $x^4+2x^3-13x^2-14x+24=0$.

2. $x^4-10x^3+35x^2-50x+24=0$.

3. $3x^4+12x^2+5x-4=0$.

4. Show that the equation $2x^7-x^4+4x^3-5=0$ has at least four imaginary roots.

5. What may be inferred respecting the roots of the equation $x^{10}-4x^6+x^4-2x-3=0$?

6. Find the least possible number of imaginary roots of the equation $x^9-x^5+x^4+x^2+1=0$.

594. Derived Functions. *To find the value of $f(x+h)$, when $f(x)$ is a rational integral function of x.*

Let $f(x)=p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n$; then

$$f(x+h)=p_0(x+h)^n+p_1(x+h)^{n-1}+p_2(x+h)^{n-2}+\cdots$$
$$+p_{n-1}(x+h)+p_n.$$

Expanding each term and arranging the result in ascending powers of h, we have

$$\begin{aligned}
&p_0x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n\\
&\quad+h\{np_0x^{n-1}+(n-1)p_1x^{n-2}+(n-2)p_2x^{n-3}+\cdots+p_{n-1}\}\\
&\quad+\frac{h^2}{\underline{|2}}\{n(n-1)p_0x^{n-2}+(n-1)(n-2)p_1x^{n-3}+\cdots+2p_{n-2}\}\\
&\quad+\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\\
&\quad+\frac{h^n}{\underline{|n}}\{n(n-1)(n-2)\cdots2\cdot1\,p_0\}.
\end{aligned}$$

This result is usually written in the form

$$f(x+h)=f(x)+hf'(x)+\frac{h^2}{\underline{|2}}f''(x)+\frac{h^3}{\underline{|3}}f'''(x)+\cdots+\frac{h^n}{\underline{|n}}f^n(x),$$

and the functions $f'(x)$, $f''(x)$, $f'''(x)$, $\cdots$ are called the *first, second, third,* $\cdots$ *derived functions* of $f(x)$.

Examining the coefficients of h, $\frac{h^2}{\underline{|2}}\cdots$, we see that *to obtain $f'(x)$ from $f(x)$ we multiply each term in $f(x)$ by the index of x in that term, and then diminish the index by unity.*

Similarly we obtain $f''(x)$, $f'''(x)$, $\cdots$.

595. Equal Roots. *If the equation $f(x)=0$ has r roots equal to a, then the equation $f'(x)=0$ will have $r-1$ roots equal to a.*

Let $\phi(x)$ be the quotient when $f(x)$ is divided by $(x-a)^r$; then $f(x)=(x-a)^r\phi(x)$.

Write $x+h$ in the place of x; thus

$$f(x+h)=(x-a+h)^r\phi(x+h);$$

$$\therefore\ f(x)+hf'(x)+\frac{h^2}{\underline{|2}}f''(x)+\cdots$$

$$=\{(x-a)^r+r(x-a)^{r-1}h+\cdots\}\{\phi(x)+h\phi'(x)+\frac{h^2}{\underline{|2}}\phi''(x)+\cdots\}.$$

In this identity, by equating the coefficients of h, we have

$$f'(x)=r(x-a)^{r-1}\phi(x)+(x-a)^r\phi'(x).$$

Thus $f'(x)$ contains the factor $x-a$ repeated $r-1$ times; that is, the equation $f'(x)=0$ has $r-1$ roots equal to a.

Similarly we may show that if the equation $f(x)=0$ has s roots equal to b, the equation $f'(x)=0$ has $s-1$ roots equal to b; and so on.

From the foregoing proof we see that if $f(x)$ contains a factor $(x-a)^r$, then $f'(x)$ contains a factor $(x-a)^{r-1}$; and thus $f(x)$ and $f'(x)$ have a common factor $(x-a)^{r-1}$. Therefore if $f(x)$ and $f'(x)$ have no common factor, no factor in $f(x)$ will be repeated; hence *the equation* $f(x)=0$ *has or has not equal roots, according as* $f(x)$ *and* $f'(x)$ *have or have not a common factor involving* x.

596. It follows that in order to obtain the equal roots of the equation $f(x)=0$, *we must first find the highest common factor of* $f(x)$ *and* $f'(x)$, *and then placing it equal to zero, solve the resulting equation.*

Ex. Solve the equation $x^4-11x^3+44x^2-76x+48=0$, which has equal roots.

Here $$f(x)=x^4-11x^3+44x^2-76x+48,$$

$$f'(x)=4x^3-33x^2+88x-76;$$

and by the ordinary rule we find that the highest common factor of $f(x)$ and $f'(x)$ is $x-2$; hence $(x-2)^2$ is a factor of $f(x)$; and

$$f(x)=(x-2)^2(x^2-7x+12)$$

$$=(x-2)^2(x-3)(x-4);$$

thus the roots are 2, 2, 3, 4.

LOCATION OF THE ROOTS.

597. If the variable x changes continuously from a to b the function $f(x)$ will change continuously from $f(a)$ to $f(b)$.

Let c and $c+h$ be any two values of x lying between a and b. We have

$$f(c+h)-f(c)=hf'(c)+\frac{h^2}{\underline{|2}}f''(c)+\cdots+\frac{h^n}{\underline{|n}}f^n(c);$$

and by taking h small enough, the difference between $f(c+h)$ and $f(c)$ can be made as small as we please; hence to a small change in the variable x there corresponds a small change in the function $f(x)$, and therefore as x changes gradually from a to b, the function $f(x)$ changes gradually from $f(a)$ to $f(b)$.

598. It is important to notice that we have not proved that $f(x)$ always increases from $f(a)$ to $f(b)$, or decreases from $f(a)$ to $f(b)$, but that it passes from one value to the other without any sudden change; sometimes it may be increasing and at other times it may be decreasing.

599. If $f(a)$ and $f(b)$ are of contrary signs then one root of the equation $f(x)=0$ must lie between a and b.

As x changes gradually from a to b, the function $f(x)$ changes gradually from $f(a)$ to $f(b)$, and therefore must pass through all intermediate values; but since $f(a)$ and $f(b)$ have contrary signs the value zero must lie between them; that is, $f(x)=0$ for some value of x between a and b.

It does not follow that $f(x)=0$ has *only one* root between a and b; neither does it follow that if $f(a)$ and $f(b)$ have the *same* sign $f(x)=0$ has no root between a and b.

600. Every equation of an odd degree has at least one real root whose sign is opposite to that of its last term.

In the function $f(x)$ substitute for x the values $+\infty$, 0, $-\infty$, successively, then

$$f(+\infty)=+\infty,\quad f(0)=p_n,\quad f(-\infty)=-\infty.$$

If p_n is positive, then $f(x)=0$ has a root lying between 0 and $-\infty$, and if p_n is negative $f(x)=0$ has a root lying between 0 and $+\infty$.

601. Every equation which is of an even degree and has its last term negative has at least two real roots, one positive and one negative.

For in this case

$$f(+\infty)=+\infty,\quad f(0)=p_n\quad f(-\infty)=+\infty;$$

but p_n is negative; hence $f(x)=0$ has a root lying between 0 and $+\infty$, and a root lying between 0 and $-\infty$.

602. If the expressions $f(a)$ and $f(b)$ have contrary signs, an odd number of roots of $f(x)=0$ will lie between a and b; and if $f(a)$ and $f(b)$ have the same sign, either no root or an even number of roots will lie between a and b.

Suppose that a is greater than b, and that $c, d, e, \cdots k$ represent all the roots of $f(x)=0$, which lie between a and b. Let $\phi(x)$ be the quotient when $f(x)$ is divided by the product $(x-c)(x-d)(x-e)\cdots(x-k)$; then

$$f(x)=(x-c)(x-d)(x-e)\cdots(x-k)\,\phi(x).$$

Hence $f(a)=(a-c)(a-d)(a-e)\cdots(a-k)\,\phi(a)$.

$$f(b)=(b-c)(b-d)(b-e)\cdots(b-k)\,\phi(b).$$

Now $\phi(a)$ and $\phi(b)$ must be of the same sign, for otherwise a root of the equation $\phi(x)=0$, and therefore of $f(x)=0$, would lie between a and b [Art. 599], which is contrary to the hypothesis. Hence if $f(a)$ and $f(b)$ have contrary signs, the expressions

$$(a-c)(a-d)(a-e)\cdots(a-k),$$
$$(b-c)(b-d)(b-e)\cdots(b-k)$$

must have contrary signs. Also the factors in the first expressions are all positive, and the factors in the second are all negative; hence the number of factors must be odd, that is, the number of roots $c, d, e, \cdots k$ must be odd.

Similarly if $f(a)$ and $f(b)$ have the same sign the number of factors must be even. In this case the given condition is satisfied if $c, d, e, \cdots k$ are all greater than a, or less than b; thus it does not necessarily follow that $f(x)=0$ has a root between a and b.

EXAMPLES XLVIII. f.

1. Find the successive derived functions of $2x^4-x^3-2x^2+5x-1$.

Solve the following equations which have equal roots:

2. $x^4-9x^2+4x+12=0$. **3.** $x^4-6x^3+12x^2-10x+3=0$.

4. $x^5-13x^4+67x^3-171x^2+216x-108=0$.

5. $x^5-x^3+4x^2-3x+2=0$.

6. $8x^4+4x^3-18x^2+11x-2=0$.

7. Show that the equation $10x^3-17x^2+x+6=0$ has a root between 0 and -1.

8. Show that the equation $x^4-5x^3+3x^2+35x-70=0$ has a root between 2 and 3, and one between -2 and -3.

9. Show that the equation $x^4-12x^2+12x-3=0$ has a root between -3 and -4, and another between 2 and 3.

10. Show that $x^5+5x^4-20x^2-19x-2=0$ has a root between 2 and 3, and a root between -4 and -5.

STURM'S THEOREM AND METHOD.

603. In 1829, Sturm, a Swiss mathematician, discovered a method of determining completely the number and situation of the real roots of an equation.

604. Let $f(x)$ be an equation from which the equal roots have been removed, and let $f_1(x)$ be the first derived function. Now divide $f(x)$ by $f_1(x)$, and denote the remainder *with its signs changed* by $f_2(x)$. Divide $f_1(x)$ by $f_2(x)$ and continue the operation, which is that of finding the H.C.F. of $f(x)$ and $f_1(x)$, except that the signs in every remainder are changed before it is used as a divisor, until a remainder is obtained independent of x; the signs in this remainder must also be changed. No other changes of sign are allowed.

The expressions $f(x)$, $f_1(x)$, $f_2(x)$, $\cdots f_n(x)$ are called *Sturm's Functions.*

Let $Q_1, Q_2, \cdots Q_{n-1}$ denote the successive quotients obtained; then the steps in the operation may be represented as follows:

$$\begin{aligned}
f(x) &= Q_1 f_1(x) - f_2(x), \\
f_1(x) &= Q_2 f_2(x) - f_3(x), \\
f_2(x) &= Q_3 f_3(x) - f_4(x), \\
&\cdots\cdots\cdots\cdots \\
f_{n-2}(x) &= Q_{n-1} f_{n-1}(x) - f_n(x).
\end{aligned}$$

From these equalities we obtain the following:

(1) Two consecutive functions cannot vanish for the same value of *x*.

For if they could, all the succeeding functions would vanish, including $f_n(x)$, which is impossible, as it is independent of x.

(2) When any function except the first vanishes for a particular value of *x*, the two adjacent functions have opposite signs.

Thus in $f_2(x) = Qf_3(x) - f_4(x)$ if $f_3(x) = 0$, we have

$$f_2(x) = -f_4(x).$$

We may now state **Sturm's Theorem.**

If in Sturm's Functions we substitute for x any particular value a and note the number of variations of sign; then assign to x a greater value b, and again note the number of variations of sign; the number of variations lost is equal to the number of real roots of $f(x)$ which lie between a and b.

(1) Let c be a value of x which makes some function *except the first* vanish; for example, $f_r(x)$, so that $f_r(c) = 0$. Now when $x = c$, $f_{r-1}(x)$, and $f_{r+1}(x)$ have contrary signs, and thus just before $x = c$ and also just after $x = c$, the three functions $f_{r-1}(x)$, $f_r(x)$, $f_{r+1}(x)$ have one permanence of sign and one variation of sign, hence no change occurs in the

number of variations when x passes through a value which makes a function except $f(x)$ vanish.

(2) Let c be a root of the equation $f(x)=0$ so that $f(c)=0$. Let h be any positive quantity.

Now $$f(c+h)=f(c)+hf_1(c)+\frac{h^2}{\lfloor 2}f_2(c)+\cdots, \quad \text{[Art. 594.]}$$

and as c is a root of the equation $f(x)=0$, $f(c)=0$, hence

$$f(c+h)=hf_1(c)+\frac{h^2}{\lfloor 2}f_2(c)+\cdots.$$

If h be taken very small, we may disregard the terms containing its higher powers and obtain

$$f(c+h)=hf_1(c),$$

and as h is a positive quantity, $f(c+h)$ and $f_1(c)$ have the same sign. That is, the function just after x passes a root has the same sign as $f_1(x)$ *at a root.*

In a like manner we may show that $f(c-h)=-hf_1(c)$, or that the function just before x passes a root has a sign opposite to $f_1(x)$ *at a root.* Thus as x increases, *Sturm's Functions* lose one variation of sign only when x passes through a root of the equation $f(x)=0$.

There is at no time a gain in the number of variations of sign, hence the theorem is established.

605. In determining the whole number of real roots of an equation $f(x)=0$ we first substitute $-\infty$ and then $+\infty$ for x in Sturm's Functions: the difference in the number of variations of sign in the two cases gives the whole number of real roots.

By substituting $-\infty$ and 0 for x we may determine the number of negative real roots, and the substitution of $+\infty$ and 0 for x gives the number of positive real roots.

606. When $+\infty$ or $-\infty$ is substituted for x, the sign of any function will be that of the highest power of x in that function.

607. Let us determine the number and situation of the real roots of $x^3 + 3x^2 - 9x - 4 = 0$.

Here $f_1(x) = 3x^2 + 6x - 9$.

Now any *positive* factor may be introduced or removed in finding $f_2(x), f_3(x)$, etc., for the *sign* of the result is not affected by so doing; hence multiplying the original equation by 3, we have

$$
\begin{array}{r|l}
3x^2+6x-9) & 3x^3+9x^2-27x-12(x+1 \\
& 3x^3+6x^2-\ \ 9x \\
\hline
& 3x^2-18x-12 \\
& 3x^2+\ \ 6x-\ \ 9 \\
\hline
& 3)-24x-\ \ 3 \\
& -\ \ 8x-\ \ 1 \qquad \therefore\ f_2(x)=8x+1.
\end{array}
$$

$$
\begin{array}{r|l}
8x+1) & 24x^2+48x-72(3x+5 \\
& 24x^2+\ \ 3x \\
\hline
& 9)45x-72 \\
& 5x-\ \ 8 \\
& \qquad 8 \\
\hline
& 40x-64 \\
& 40x+\ \ 5 \\
\hline
& -69 \qquad \therefore\ f_3(x)=69.
\end{array}
$$

We therefore have

$$
\begin{aligned}
f(x) &= x^3 + 3x^2 - 9x - 4, \\
f_1(x) &= 3x^2 + 6x - 9, \\
f_2(x) &= 8x + 1, \\
f_3(x) &= 69.
\end{aligned}
$$

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	
When $x = -\infty$ we have	$-$	$+$	$-$	$+$	3 variations.
When $x = +\infty$ we have	$+$	$+$	$+$	$+$	no variations.
When $x = 0$ we have	$-$	$-$	$+$	$+$	1 variation.

Hence the number of real roots is 3, of which one is positive and two are negative.

To determine the situation of these roots we substitute different numbers, commencing at 0 and working in each direction, thus:

	$f(x)$	$f_1(x)$	$f_2(x)$	$f_3(x)$	
$x = -\infty$	$-$	$+$	$-$	$+$	3 variations.
$x = -5$	$-$	$+$	$-$	$+$	3 variations.
$x = -4$	$+$	$+$	$-$	$+$	2 variations.
$x = -3$	$+$	$\pm$	$-$	$+$	2 variations.
$x = -2$	$+$	$-$	$-$	$+$	2 variations.
$x = -1$	$+$	$-$	$-$	$+$	2 variations.
$x = 0$	$-$	$-$	$+$	$+$	1 variation.
$x = 1$	$-$	$\pm$	$+$	$+$	1 variation.
$x = 2$	$-$	$+$	$+$	$+$	1 variation.
$x = 3$	$+$	$+$	$+$	$+$	no variation.
$x = \infty$	$+$	$+$	$+$	$+$	no variation.

Thus one root lies between -4 and -5; a second lies between 0 and -1, and the third lies between 2 and 3.

EXAMPLES XLVIII. g.

Determine the number and situation of the real roots of:

1. $x^3 - 4x^2 - 6x + 8 = 0$.
2. $2x^4 - 11x^2 + 8x - 16 = 0$.
3. $x^3 - 7x + 7 = 0$.
4. $x^4 - 4x^3 + 6x^2 - 12x + 2 = 0$.
5. $x^4 - 4x^3 + x^2 + 6x + 2 = 0$.
6. $x^4 - x^3 + x - 1 = 0$.
7. $x^3 - 9x^2 + 23x - 16 = 0$.
8. $x^5 + x^3 - 2x^2 + 2x - 1 = 0$.

GRAPHICAL REPRESENTATION OF FUNCTIONS.

Coördinates.

608. Two lines drawn at right angles to each other as in Fig. 1 form a simple system of lines of reference. Their intersection, O, is called the *origin*. Distances from O along XX' are called *abscissas*; distances from XX' on a line parallel to YY' are called *ordinates*.

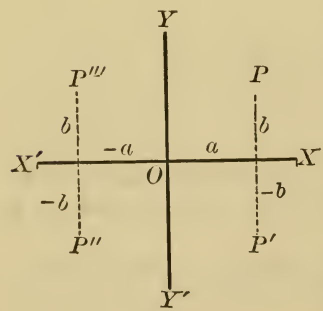


Fig. 1.

609. Abscissas measured to the *right* of the origin are considered *positive*, and to the *left*, *negative*. Ordinates measured *above* XX' are considered *positive*, and when taken *below* XX' are *negative*.

610. The abscissa and ordinate of a point are called the *co-ordinates* of that point, and the lines XX' and YY' are called Co-ordinate Axes, Axis of X, and Axis of Y, or Axis of Abscissas, and Axis of Ordinates.

611. *Any* point in the plane can be given by means of these co-ordinates: thus the point P of Fig. 1 is located by measuring the distance a to the *right* of O on the axis of abscissas, and then taking a distance b vertically upwards.

Since a and b can be either positive or negative, a point P' is found by taking a positive and b negative; P'' is found by taking a negative and b negative; and P''' is found by taking a negative and b positive.

Abscissas and ordinates are generally represented by x and y respectively. Thus for the point P, $x = a$, and $y = b$; for P', $x = a$, and $y = -b$, etc.

612. Instead of writing "the point whose co-ordinates are 5 and 3," a more concise form is used: thus the point (5, 3) means that the point will be found by taking an abscissa of 5 units and an ordinate of 3.

Locate the points $(3, -2)$; $(5, 8)$; $(-4, 4)$; $(-8, -3)$.

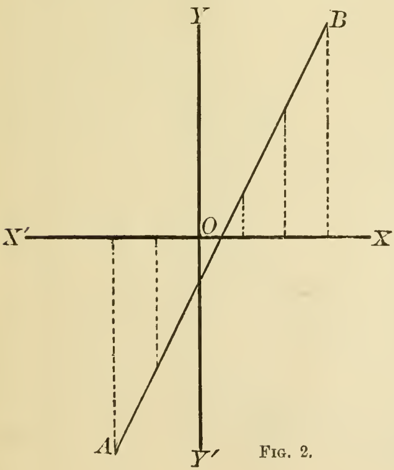


FIG. 2.

GRAPH OF A FUNCTION.

613. Let $f(x)$ be any rational integral function of x, and let us place it equal to y. If we give a series of numerical values to x we can obtain corresponding values for y. Now laying off the values of x as abscissas, and the corresponding values of y as ordinates, we have a series of points which lie upon a line called the *graph* of the given function.

Ex. **1**. Construct the graph of $2x - 1$.

Let $2x - 1 = y$. Giving to x successive values, we obtain the corresponding values of y as follows:

$x = -2,\quad y = -5.$
$x = -1,\quad y = -3.$
$x = 0,\quad y = -1.$
$x = 1,\quad y = 1.$
$x = 2,\quad y = 3.$
$x = 3,\quad y = 5.$

Locating these points as explained in Art. 611 and drawing a line through them, we have in this case the straight line AB, Fig. 2, as the *graph* required.

Ex. **2**. Plot the equation $x^2 - 2x - 4$.

Putting $y = x^2 - 2x - 4$, we obtain the following values:

$x = -2,\quad y = 4.$
$x = -1,\quad y = -1.$
$x = 0,\quad y = -4.$
$x = 1,\quad y = -5.$
$x = 2,\quad y = -4.$
$x = 3,\quad y = -1.$
$x = 4,\quad y = 4.$

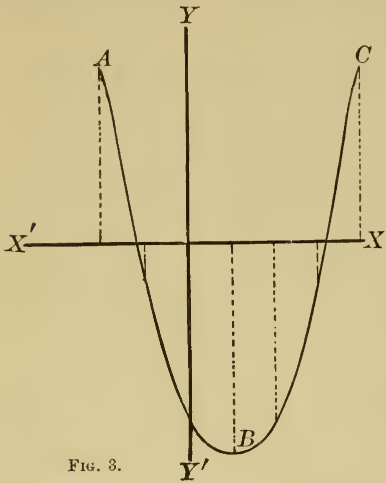


Fig. 3.

The points lie on the line ABC, Fig. 3, which is the *graph* of the given equation.

Ex. **3**. Plot the *graph* of $x^3 - 2x$.

Assuming $y = x^3 - 2x$, we have

$x = -2,\quad y = -4.$
$x = -1,\quad y = 1.$
$x = 0,\quad y = 0.$
$x = 1,\quad y = -1.$
$x = 2,\quad y = 4.$

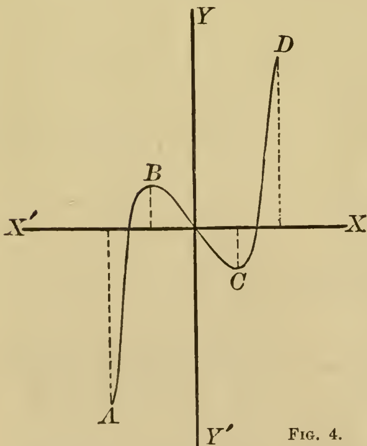


Fig. 4.

The line $ABCD$, Fig. 4, is the required *graph*.

By taking other values between those assumed, we may locate the curve with greater precision.

614. In giving values to x it is evident that the substitution of a root gives $y=0$, that is, the ordinate, or distance from the Axis of Abscissas, is 0; hence where the graph cuts the Axis of X we have the location of a real root, and the graph will cross the Axis of X as many times as the equation has real and unequal roots. If the roots of the equation be imaginary, the curve will not touch the Axis of X.

EXAMPLES XLVIII. h.

Construct the graphs of the following functions:

1. $2x-3$. **3.** x^2-5. **5.** x^3-2. **7.** x^4-3x^2+3.

2. x^2+2x+1. **4.** x^3+x-1. **6.** x^3-5x+3. **8.** $x^3+3x^2+5x-12$.

SOLUTION OF HIGHER NUMERICAL EQUATIONS.

Commensurable Roots.

615. A real root which is either an integer or a fraction is said to be **commensurable.**

By Art. 582 we can transform an equation with fractional coefficients into another which has all of its coefficients integers, that of the first term being unity: hence we need consider only equations of this form. Such equations cannot have for a root a rational fraction in its lowest terms [Art. 575], therefore we have only to find the integral roots. By Art. 570 the last term of $f(x)$ is divisible by every integral root, therefore to find the commensurable roots of $f(x)$ it is only necessary *to find the integral divisors of the last term and determine by trial which of them are roots.*

616. Newton's Method. If the divisors are small numbers we may readily ascertain by actual substitution whether they are roots. In other cases we may use the method of Arts. 562 and 566 or the *Method of Divisors*, sometimes called *Newton's Method.*

Suppose a to be an integral root of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\cdots+p_{n-1}x+p_n=0.$$

By substitution we have

$$a^n + p_1 a^{n-1} + p_2 a^{n-2} + \cdots + p_{n-1} a + p_n = 0.$$

Transposing and dividing throughout by a, we obtain

$$\frac{p_n}{a} = -p_{n-1} - \cdots - p_2 a^{n-3} - p_1 a^{n-2} - a^{n-1},$$

in which it is evident that $\frac{p_n}{a}$ must be an integer. Denoting $\frac{p_n}{a}$ by Q and transposing $-p_{n-1}$,

$$Q + p_{n-1} = - \cdots - p_2 a^{n-3} - p_1 a^{n-2} - a^{n-1}.$$

Dividing again by a gives

$$\frac{Q + p_{n-1}}{a} = - \cdots - p_2 a^{n-4} - p_1 a^{n-3} - a^{n-2}.$$

Again, as before, the first member of the equation must be an integer. Denoting it by Q_2 and proceeding as before, we must after n divisions obtain a result

$$\frac{Q_{n-1} + p_1}{a} = -1.$$

Hence if a represents one of the integral divisors of the last term we have the following rule:

Divide the last term by a and add the coefficient of x to the quotient.

Divide this sum by a, and if the quotient is an integer add to it the coefficient of x^2.

Proceed in this manner, and if a is a root of the equation each quotient will be an integer and the last quotient will be -1.

The advantage of Newton's method is that the obtaining of a fractional quotient at any point of the division shows at once that the divisor is not a root of the equation.

Ex. Find the integral roots of $x^4 + 4x^3 - x^2 - 16x - 12 = 0$. By Descartes' Rule the equation cannot have more than one positive root, nor more than three negative roots.

The integral divisors of -12 are ± 1, ± 2, ± 3, ± 4, ± 6. Substitution shows that -1 is a root, and that $+1$ is not a root.

To ascertain if 2 is a root, arrange the work as follows:

$$\begin{array}{rrrrr|l} 1 & +4 & -1 & -16 & -12 & \underline{2} \\ & -1 & -6 & -11 & -6 & \\ \hline & & -2 & -12 & -22 & \end{array}$$

Hence 2 is a root.

EXPLANATION. The first line contains the coefficients of the original equation, and the divisor 2. Dividing the *last* term, -12, by 2 gives a quotient -6; adding -16, the coefficient of x, gives -22. Dividing -22 by 2 gives -11; adding -1, the coefficient of x^2, gives -12. Dividing -12 by 2 gives -6; adding $+4$, the coefficient of x^3, gives -2, which divided by 2 gives a final quotient of -1, hence 2 is a root.

Since the equation can have no more than one positive root, we will only make trial of the remaining negative divisors, thus:

$$\begin{array}{rrrrr|l} 1 & +4 & -1 & -16 & -12 & \underline{-6} \\ & & & & +2 & \\ \hline & & & & -14 & \end{array}$$

Hence -6 is not a root.

$$\begin{array}{rrrrr|l} 1 & +4 & -1 & -16 & -12 & \underline{-4} \\ & & & & +3 & \\ \hline & & & & -13 & \end{array}$$

Hence -4 is not a root.

$$\begin{array}{rrrrr|l} 1 & +4 & -1 & -16 & -12 & \underline{-3} \\ -1 & -1 & +4 & +4 & & \\ \hline & +3 & +3 & -12 & & \end{array}$$

Hence -3 is a root.

$$\begin{array}{rrrrr|l} 1 & +4 & -1 & -16 & -12 & \underline{-2} \\ -1 & -2 & +5 & +6 & & \\ \hline & +2 & +4 & -10 & & \end{array}$$

Hence -2 is a root.

EXAMPLES XLVIII. i.

Solve the following equations, which have one or more integral roots:

1. $x^3 - 9x^2 + 26x - 24 = 0$.
2. $x^3 - x - 2x^2 + 2 = 0$.
3. $2x^3 + 5x^2 - 11x + 4 = 0$.
4. $4x^3 - 16x^2 + 31x - 14 = 0$.
5. $x^3 - 2x^2 - 29x + 30 = 0$.
6. $x^3 - 8x^2 + 5x + 14 = 0$.
7. $x^4 - 2x^3 - 7x^2 + 8x + 12 = 0$.
8. $x^4 - 4x^3 - 14x^2 + 36x + 45 = 0$.
9. $x^4 - 3x^2 - 42x - 40 = 0$.
10. $x^4 - 10x^2 - 20x - 16 = 0$.
11. $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0$.
12. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$.
13. $x^4 - 3x^2 - 6x - 2 = 0$.
14. $x^4 - 2x^3 - 12x^2 + 10x + 3 = 0$.
15. $6x^4 - x^3 - 17x^2 + 16x - 4 = 0$.
16. $x^4 - 2x^3 - 13x^2 + 14x + 24 = 0$.
17. $x^4 + 4x^3 - 22x^2 - 4x + 21 = 0$.
18. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$.
19. $x^4 - 6x^2 - 16x + 21 = 0$.
20. $x^4 + 4x^3 - x^2 - 16x - 12 = 0$.
21. $2x^4 - x^3 - 29x^2 + 34x + 24 = 0$.
22. $x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12 = 0$.

617. The Cube Roots of Unity.

Suppose $x = \sqrt[3]{1}$; then $x^3 = 1$, or $x^3 - 1 = 0$;

that is $$(x-1)(x^2+x+1) = 0.$$

$\therefore$ either $x - 1 = 0$, or $x^2 + x + 1 = 0$;

whence $$x = 1, \text{ or } x = \frac{-1 \pm \sqrt{-3}}{2}.$$

It may be shown by actual involution that each of these values when cubed is equal to unity. Thus unity has three cube roots,

$$1, \frac{-1+\sqrt{-3}}{2}, \frac{-1-\sqrt{-3}}{2};$$

two of which are imaginary expressions.

Let us denote these by a and b; then since they are the roots of the equation

$$x^2 + x + 1 = 0,$$

their product is equal to unity;

that is, $$ab = 1;$$

$$\therefore a^3 b = a^2;$$

that is, $$b = a^2, \text{ since } a^3 = 1.$$

Similarly we may show that $a = b^2$.

618. Since *each of the imaginary roots is the square of the other*, it is usual to denote the three cube roots of unity by 1, ω, ω^2.*

Also ω satisfies the equation $x^2 + x + 1 = 0$;

$$\therefore 1 + \omega + \omega^2 = 0;$$

that is, *the sum of the three cube roots of unity is zero.*

Again $$\omega \cdot \omega^2 = \omega^3 = 1;$$

therefore (1) *the product of the two imaginary roots is unity;*

(2) *every integral power of ω^3 is unity.*

* The Greek letter Omega.

CARDAN'S METHOD FOR THE SOLUTION OF CUBIC EQUATIONS.

619. The general type of a cubic equation is

$$x^3 + Px^2 + Qx + R = 0,$$

but as explained in Art. 585 this equation can be reduced to the simpler form $\qquad x^3 + qx + r = 0,$

which we shall take as the *standard form* of a cubic equation.

620. We proceed to solve the equation $x^3 + qx + r = 0$.

Let $x = y + z$; then

$$x^3 = y^3 + z^3 + 3\,yz\,(y + z) = y^3 + z^3 + 3\,yzx,$$

and the given equation becomes

$$y^3 + z^3 + (3\,yz + q)\,x + r = 0.$$

At present y, z are any two quantities subject to the condition that their sum is equal to one of the roots of the given equation; if we further suppose that they satisfy the equation $3\,yz + q = 0$, they are completely determinate. We thus obtain

$$y^3 + z^3 = -r \quad \ldots\ldots \quad (1),$$

$$z^3 = -\frac{q^3}{27\,y^3} \quad \ldots\ldots \quad (2);$$

hence $\qquad g^3 - \dfrac{q^3}{27\,y^3} = -r$, or $y^6 + ry^3 = \dfrac{q^3}{27}$.

Solving this equation,

$$y^3 = -\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}} \quad \ldots\ldots \quad (3).$$

Substituting in (1), $\quad z^3 = -\dfrac{r}{2} - \sqrt{\dfrac{r^2}{4} + \dfrac{q^3}{27}} \quad \ldots\ldots \quad (4).$

We obtain the value of x from the relation $x = y + z$; thus

$$x = \left\{-\frac{r}{2} + \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right\}^{\frac{1}{3}} + \left\{-\frac{r}{2} - \sqrt{\frac{r^2}{4} + \frac{q^3}{27}}\right\}^{\frac{1}{3}} \quad . \quad (5).$$

The above solution is generally known as *Cardan's Solution*, as it was first published by him in the *Ars Magna*, in 1545. Cardan obtained the solution from Tartaglia; but the solution of the cubic seems to have been due originally to Scipio Ferreo, about 1505.

In this solution we assume $x = y + z$, and from (2) find $z = -\dfrac{q}{3y}$, hence *to solve a cubic equation of the form*

$$x^3 + qx + r = 0$$

we substitute $y - \dfrac{q}{3y}$ *for* x.

Ex. Solve the equation $x^3 - 15x = 126$.

Put $y - \left(\dfrac{-15}{3y}\right)$ or $y + \dfrac{5}{y}$ for x, then

$$y^3 + 15y + \frac{75}{y} + \frac{125}{y^3} - 15y - \frac{75}{y} = 126,$$

or $$y^3 + \frac{125}{y^3} = 126,$$

whence $$y^6 - 126y^3 = -125.$$

$$\therefore\ y^3 = 125,$$

$$\therefore\ y = 5.$$

But $x = y + \dfrac{5}{y} = 6$.

Dividing the given equation $x^3 - 15x - 126 = 0$ by $x - 6$, we obtain the depressed equation

$$x^2 + 6x + 21 = 0,$$

the roots of which are $-3 + 2\sqrt{-3}$, and $-3 - 2\sqrt{-3}$.

Thus the roots of $x^3 - 15x = 126$ are 6, $-3 + 2\sqrt{-3}$, and $-3 - 2\sqrt{-3}$.

BIQUADRATIC EQUATIONS.

621. We shall now give a brief discussion of some of the methods which are employed to obtain the general solution of a biquadratic equation. It will be found that in each of the methods we have first to solve an auxiliary cubic equation; and thus it will be seen that as in the case of the cubic, the general solution is not adapted for writing down the solution of a given numerical equation.

622. The solution of a biquadratic equation was first obtained by Ferrari, a pupil of Cardan, as follows:

Denote the equation by

$$x^4 + 2px^3 + qx^2 + 2rx + s = 0;$$

add to each side $(ax + b)^2$, the quantities a and b being determined so as to make the left side a perfect square; then

$$x^4 + 2px^3 + (q + a^2)x^2 + 2(r + ab)x + s + b^2 = (ax + b)^2.$$

Suppose that the left side of the equation is equal to $(x^2 + px + k)^2$; then by comparing the coefficients, we have

$$p^2 + 2k = q + a^2, \; pk = r + ab, \; k^2 = s + b^2;$$

by eliminating a and b from these equations, we obtain

$$(pk - r)^2 = (2k + p^2 - q)(k^2 - s),$$

or $$2k^3 - qk^2 + 2(pr - s)k + p^2s - qs - r^2 = 0.$$

From this cubic equation one real value of k can always be found [Art. 600]; thus a and b are known. Also

$$(x^2 + px + k)^2 = (ax + b)^2;$$

$$\therefore\; x^2 + px + k = \pm (ax + b);$$

and the values of x are to be obtained from the two quadratics

$$x^2 + (p - a)x + (k - b) = 0,$$

and $$x^2 + (p + a)x + (k + b) = 0.$$

Ex. Solve the equation

$$x^4 - 2x^3 - 5x^2 + 10x - 3 = 0.$$

Add $a^2x^2 + 2abx + b^2$ to each side of the equation, and assume

$$x^4 - 2x^3 + (a^2 - 5)x^2 + 2(ab + 5)x + b^2 - 3 = (x^2 - x + k)^2;$$

then by equating coefficients, we have

$$a^2 = 2k + 6, \; ab = -k - 5, \; b^2 = k^2 + 3;$$

$$\therefore\; (2k + 6)(k^2 + 3) = (k + 5)^2;$$

$$\therefore\; 2k^3 + 5k^2 - 4k - 7 = 0.$$

By trial, we find that $k = -1$; hence $a^2 = 4$, $b^2 = 4$, $ab = -4$.

But from the assumption, it follows that

$$(x^2 - x + k)^2 = (ax + b)^2.$$

Substituting the values of k, a and b, we have the two equations

$$x^2 - x - 1 = \pm (2x - 2);$$

that is $\quad x^2 - 3x + 1 = 0$, and $x^2 + x - 3 = 0$;

whence the roots are $\quad \dfrac{3 \pm \sqrt{5}}{2}, \quad \dfrac{-1 \pm \sqrt{13}}{2}.$

623. The following solution was given by Descartes in 1637.

Suppose that the biquadratic equation is reduced to the form

$$x^4 + qx^2 + rx + s = 0;$$

assume $\; x^4 + qx^2 + rx + s = (x^2 + kx + l)(x^2 - kx + m);$

then by equating coefficients, we have

$$l + m - k^2 = q, \; k(m - l) = r, \; lm = s.$$

From the first two of these equations, we obtain

$$2m = k^2 + q + \frac{r}{k}, \; 2l = k^2 + q - \frac{r}{k};$$

hence substituting in the third equation,

$$(k^3 + qk + r)(k^3 + qk - r) = 4sk^2,$$

or $$k^6 + 2qk^4 + (q^2 - 4s)k^2 - r^2 = 0.$$

This is a cubic in k^2 which always has one real positive solution [Art. 600]; thus when k^2 is known the values of l and m are determined, and the solution of the biquadratic is obtained by solving the two quadratics

$$x^2 + kx + l = 0, \text{ and } x^2 - kx + m = 0.$$

Ex. Solve the equation

$$x^4 - 2x^2 + 8x - 3 = 0.$$

Assume $\quad x^4 - 2x^2 + 8x - 3 = (x^2 + kx + l)(x^2 - kx + m);$

then by equating coefficients, we have

$$l + m - k^2 = -2, \; k(m - l) = 8, \; lm = -3;$$

whence we obtain $\quad (k^3 - 2k + 8)(k^3 - 2k - 8) = -12k^2,$

or $$k^6 - 4k^4 + 16k^2 - 64 = 0.$$

This equation is clearly satisfied when $k^2 - 4 = 0$, or $k = \pm 2$. It will be sufficient to consider one of the values of k; putting $k = 2$, we have

$$m + l = 2, \quad m - l = 4; \text{ that is, } l = -1, \quad m = 3.$$

Thus $\quad x^4 - 2x^2 + 8x - 3 = (x^2 + 2x - 1)(x^2 - 2x + 3);$

hence $\quad x^2 + 2x - 1 = 0$, and $x^2 - 2x + 3 = 0$;

and therefore the roots are $-1 \pm \sqrt{2}$, $1 \pm \sqrt{-2}$.

624. The general algebraic solution of equations of a degree higher than the fourth has not been obtained, and Abel's demonstration of the impossibility of such a solution is generally accepted by mathematicians. If, however, the coefficients of an equation are numerical, the value of any real root may be found to any required degree of accuracy by the method of Art. 626.

EXAMPLES XLVIII. k.

Solve the following equations:

1. $x^3 - 18x = 35.$

2. $x^3 + 72x - 1720 = 0.$

3. $x^3 + 63x - 316 = 0.$

4. $x^3 + 21x + 342 = 0.$

5. $28x^3 - 9x^2 + 1 = 0.$

6. $x^3 - 15x^2 - 33x + 847 = 0.$

7. $2x^3 + 3x^2 + 3x + 1 = 0.$

8. $x^3 - 6x^2 + 3x - 18 = 0.$

9. $8x^3 - 36x + 27 = 0.$

10. $x^3 - 15x - 4 = 0.$

11. $x^4 + 8x^3 + 9x^2 - 8x - 10 = 0.$

12. $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0.$

13. $x^4 - 3x^2 - 6x - 2 = 0.$

14. $x^4 - 2x^3 - 12x^2 + 10x + 3 = 0.$

INCOMMENSURABLE ROOTS.

625. The incommensurable roots of an equation cannot be found exactly. If, however, a sufficient number of the initial figures of the root have been found to distinguish it from the other roots we may carry the approximation to the exact value to any required degree of accuracy by a method first published in 1819 by W. G. Horner.

HORNER'S METHOD OF APPROXIMATION.

626. Let it be required to solve the equation

$$x^3 - 3x^2 - 2x + 5 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

By Sturm's Theorem there are 3 real roots and one of them lies between 1 and 2; we will find its value to four places of decimals, which will sufficiently illustrate the method.

Diminishing the roots of the equation by 1 [Arts. 583, 584], we have

$$\begin{array}{ccccc}
1 & -3 & -2 & +5 & \underline{\lfloor 1} \\
 & \underline{1} & \underline{-2} & \underline{-4} & \\
 & -2 & -4 & 1 & \\
 & \underline{1} & \underline{-1} & & \\
 & -1 & -5 & & \\
 & \underline{1} & & & \\
 & 0 & & &
\end{array}$$

The transformed equation is

$$y^3 - 5y + 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

Equation (1) has a root between 1 and 2. The roots of equation (2) are each less by 1 than those of equation (1); hence equation (2) has a root between 0 and 1. This root being less than unity the higher powers of y are each less than y. Neglecting them, we obtain an approximate value of y from $-5y + 1 = 0$, or $y = .2$.

Diminishing the roots of (2), the first transformed equation, by .2, we have

$$\begin{array}{ccccc}
1 & \pm 0 & -5 & +1 & \underline{\lfloor .2} \\
 & \underline{.2} & \underline{.04} & \underline{.992} & \\
 & .2 & -4.96 & .008 & \\
 & \underline{.2} & \underline{.08} & & \\
 & .4 & -4.88 & & \\
 & \underline{.2} & & & \\
 & .6 & & &
\end{array}$$

The transformed equation is

$$z^3 + .6\,z^2 - 4.88\,z + .008 \quad . \quad . \quad . \quad . \qquad (3).$$

Equation (2) has a root between .2 and .3; the roots of equation (3) are less by .2 than those of equation (2); hence equation (3) has a root between 0 and .1. Neglecting in equation (3) the terms involving the higher powers, as was done in the case of the first transformed equation, we have

$$-4.88\,z + .008 = 0, \text{ or } z = .001.$$

Diminishing the roots of (3), the second transformed equation, by .001, we have

$$\begin{array}{rrrrl}
1 & +.6 & -4.88 & +.008 & \underline{|.001} \\
 & .001 & .000601 & -.004879399 & \\
\cline{2-4}
 & .601 & -4.879399 & .003120601 & \\
 & .001 & .000602 & & \\
\cline{2-3}
 & .602 & -4.878797 & & \\
 & .001 & & & \\
\cline{2-2}
 & .603 & & &
\end{array}$$

The transformed equation is

$$v^3 + .603\,v^2 - 4.878797\,v + .003120601 \quad . \quad . \quad . \quad . \qquad (4).$$

Equation (3) has a root between .001 and .002; the roots of equation (4) are less than those of equation (3) by .001; therefore equation (4) has a root between 0 and .001. Neglecting the terms involving v^3 and v^4, and solving

$$-4.878797\,v + .003120601 = 0,$$

we have $v = .0006$. Diminishing the roots of (4), the third transformed equation, by .0006, we find this to be the correct figure for the fourth decimal place; hence 1.2016 is the value, to the fourth decimal place, of the root which lies between 1 and 2.

Denoting the coefficients of the successive transformed equations by (A), (B), (C), etc., the work is more compactly arranged thus:

1	-3	-2	$+5$	1.2016
	1	-2	-4	
	-2	-4	(A) 1	
	1	-1	$-\ .992$	
	-1	(A) -5	(B) $.008$	
	1	$.04$	$-\ .004879399$	
	(A) 0	-4.96	(C) $.003120601$	
	$.2$	$.08$		
	$.2$	(B) -4.88		
	$.2$	$.000601$		
	$.4$	-4.879399		
	$.2$	$.000602$		
	(B) $.6$	(C) -4.878797		
	$.001$			
	$.601$			
	$.001$			
	$.602$			
	$.001$			
	(C) $.603$			

We may now state the rule for finding the approximate value of a positive incommensurable root by Horner's Method.

Rule. *Find the integral part of the root by Sturm's Theorem or method of Art.* 602.

Transform the equation into another, each of whose roots shall be less by the integral part of the root.

If in this transformed equation the coefficient of the first power of the unknown quantity and the last term have the same sign, another figure of the root should be found by the method used to find the integral part of the root. If, however, the signs of these terms are unlike, divide the latter by the former, and the first figure of the quotient will be, approximately, the next figure of the root.

Transform the last equation into another whose roots shall be less by this approximate figure of the root found by division, and proceed as before to find another figure of the root.

627. It sometimes happens that the division of the last term of the first transformed equation by the coefficient of x in that equation gives a quotient greater than unity. In that case, as where the signs of these terms are alike, we obtain another figure of the root by the method used to obtain the integral part of the root.

628. If in any transformed equation *after the first* the signs of the last two terms are *the same*, the figure of the root used in making the transformation is too large and must be diminished until these terms have unlike signs.

629. If in any transformed equation the coefficient of the first power of the unknown quantity is zero, we may obtain the next figure of the root by using the *coefficient of the second power of the unknown quantity as a divisor and taking the square root of the result.*

630. *Negative incommensurable roots* may be found by transforming the equation into one whose roots shall be positive [Art. 580], and finding the corresponding root. This result with its sign changed will be the root required.

631. Any Root of Any Number. By Horner's Method we can find approximately any root of any number; for placing $\sqrt[n]{a}$ equal to x we have for solution the equation $x^n = a$, or $x^n - a = 0$.

EXAMPLES XLVIII. 1.

Compute the root which is situated between the given limits in the following equations:

1. $x^3 + 10x^2 + 6x - 120 = 0$; root between 2 and 3.
2. $x^3 - 2x - 5 = 0$; root between 2 and 3.
3. $x^4 - 2x^3 + 21x - 23 = 0$; root between 1 and 2.
4. $x^3 + x - 1000 = 0$; root between 9 and 10.
5. $x^3 + x^2 + x - 100 = 0$; root between 4 and 5.
6. $2x^3 + 3x^2 - 4x - 10 = 0$; root between 1 and 2.
7. $x^3 - 46x^2 - 36x + 18 = 0$; root between 0 and 1.

8. $x^3 + x - 3 = 0$; root between 1 and 2.

9. $x^3 + 2x - 20 = 0$; root between 2 and 3.

10. $x^3 + 10x^2 + 8x - 120 = 0$; root between 2 and 3.

11. $3x^3 + 5x - 40 = 0$; root between 2 and 3.

12. $x^4 - 12x^2 + 12x - 3 = 0$; root between -3 and -4.

13. $x^5 - 4x^4 + 7x^3 - 863 = 0$; root between 4 and 5.

Find the real roots of the following equations:

14. $x^3 - 3x - 1 = 0$.

15. $x^3 - 22x - 24 = 0$.

16. $x^4 - 8x^3 + 12x^2 + 4x - 8 = 0$.

17. $x^4 + x^3 + x^2 + 3x - 100 = 0$.

Find to four decimals, by Horner's Method, the value of the following:

18. $\sqrt[3]{11}$. 19. $\sqrt[4]{13}$. 20. $\sqrt[5]{5}$. 21. $\sqrt[6]{7}$.

MISCELLANEOUS EXAMPLES VII.

1. Simplify $b - \{b - (a + b) - [b - (b - \overline{a - b})] + 2a\}$.

2. Find the sum of

$a + b - 2(c + d)$, $b + c - 3(d + a)$ and $c + d - 4(a + b)$.

3. Multiply $\frac{1}{2}x + \frac{2}{3}y$ by $x - \frac{1}{3}y$.

4. If $x = 6$, $y = 4$, $z = 3$, find the value of $\sqrt[3]{2x + 3y + z}$.

5. Find the square of $2 - 3x + x^2$.

6. Solve $\dfrac{x+3}{x-1} + \dfrac{x-4}{x-6} = 2$.

7. Find the H.C.F. of $a^3 - 2a - 4$ and $a^3 - a^2 - 4$.

8. Simplify $\dfrac{2a}{a+b} + \dfrac{2b}{a-b} - \dfrac{a^2+b^2}{a^2-b^2}$.

9. Solve $\left.\begin{aligned} \frac{3}{5}x + \frac{y}{4} &= 13 \\ \frac{1}{3}x - \frac{y}{8} &= 3 \end{aligned}\right\}$.

10. Two digits, which form a number, change places when 18 is added to the number, and the sum of the two numbers thus formed is 44: find the digits.

11. If $a = 1$, $b = -2$, $c = 3$, $d = -4$, find the value of

$$\frac{a^2b^2 + b^2c + d(a - b)}{10a - (c + b)^2}.$$

12. Subtract $-x^2+y^2-z^2$ from the sum of

$$\tfrac{1}{3}x^2+\tfrac{1}{4}y^2,\ \ \tfrac{1}{5}y^2+\tfrac{1}{3}z^2,\ \text{and}\ \tfrac{1}{3}z^2-\tfrac{1}{4}x^2.$$

13. Write the cube of $x+8y$.

14. Simplify $\dfrac{x^2+xy}{x^2+y^2}\times\dfrac{x^4-y^4}{xy+y^2}\times\dfrac{y}{x}$.

15. Solve $\tfrac{3}{5}(2x-7)-\tfrac{2}{3}(x-8)=\dfrac{4x+1}{15}+4$.

16. Find the H. C. F. and L. C. M. of

$$x^4+x^3+2x-4\ \text{and}\ x^3+3x^2-4.$$

17. Find the square root of $4a^4+9(1-2a)+3a^2(7-4a)$.

18. Solve $\left.\begin{aligned} y&=\dfrac{x+a}{2}+\dfrac{b}{3}\\ x&=\dfrac{y+b}{2}+\dfrac{a}{3}\end{aligned}\right\}.$

19. Simplify $\left(\dfrac{a}{x+a}-\dfrac{x}{x-a}\right)\div\dfrac{x^2+a^2}{x^2+ax}$.

20. When 1 is added to the numerator and denominator of a certain fraction the result is equal to $\tfrac{3}{2}$; and when 1 is subtracted from its numerator and denominator, the result is equal to 2: find the fraction.

21. Show that the sum of $12a+6b-c$, $-7a-b+c$ and $a+b+6c$, is six times the sum of $25a+13b-8c$, $-13a-13b-c$, and $-11a+b+10c$.

22. Divide $x^2-xy+\tfrac{3}{16}y^2$ by $x-\tfrac{1}{4}y$.

23. Add together $18\left\{\dfrac{2x}{9}-\dfrac{1}{6}\left(\dfrac{2y}{3}+z\right)\right\}$,

$$24\left(\frac{3x}{8}-\frac{2y-3z}{12}\right),\ \text{and}\ 30\left\{\frac{7z}{15}-\frac{4}{5}(2x-y)\right\}.$$

24. Find the factors of (i.) $10x^2+79x-8$. (ii.) $729x^6-y^6$.

25. Solve $\dfrac{2x-1}{5}+\dfrac{5x+3}{17}=3-\dfrac{4x-118}{11}$.

26. Find the value of

$$(5a-3b)(a-b)-b\{3a-c(4a-b)-b^2(a+c)\},$$

when $a=0,\ b=-1,\ c=\tfrac{1}{2}$.

27. Find the H. C. F. of $7x^3-10x^2-7x+10$ and $2x^3-x^2-2x+1$.

28. Simplify $\dfrac{x^2-7xy+12y^2}{x^2+5xy+6y^2}\div\dfrac{x^2-5xy+4y^2}{x^2+xy-2y^2}$.

29. Solve $\left.\begin{aligned} 3\,abx + \quad y &= 9\,b \\ 4\,abx + 3\,y &= 17\,b \end{aligned}\right\}.$

30. Find the two times between 7 and 8 o'clock when the hands of a watch are separated by 15 minutes.

31. If $a = 1,\ b = -2,\ c = 3,\ d = -4,$ find the value of

$$\sqrt{d^2 - 4\,b + a^2} - \sqrt{c^3 + b^3 + a + d}.$$

32. Multiply the product of $\frac{1}{4}x^2 - \frac{1}{2}xy + y^2$ and $\frac{1}{2}x + y$ by $x^3 - 8\,y^3$.

33. Simplify by removing brackets $a^4 - \{4\,a^3 - (6\,a^2 - 4\,a + 1)\}$
$-[-2 - \{a^4 - (-4\,a^3 - \overline{6\,a^2 - 4\,a})\} - (8\,a - 1)]$.

34. Find the remainder when $5\,x^4 - 7\,x^3 + 3\,x^2 - x + 8$ is divided by $x - 4$.

35. Simplify $\dfrac{x^2 + y^2}{x^2 - xy} \times \dfrac{xy - y^2}{x^4 - y^4} \times \dfrac{x}{y}$.

36. Solve $\left.\begin{aligned} \frac{x - 11}{3} + y &= 18 \\ 2\,x + \frac{y - 13}{4} &= 29 \end{aligned}\right\}.$

37. Find the square root of $4\,x^6 - 12\,x^4 + 28\,x^3 + 9\,x^2 - 42\,x + 49$.

38. Solve $.006\,x - .491 + .723\,x = -.005$.

39. Find the L. C. M. of $x^3 + y^3$, $3\,x^2 + 2\,xy - y^2$, and $x^3 - x^2y + xy^2$.

40. A bill of $12.50 is paid with quarters and half-dollars, and twice the number of half-dollars exceeds three times that of the quarters by 10: how many of each are used?

41. Simplify $(a+b+c)^2 - (a-b+c)^2 + (a+b-c)^2 - (-a+b+c)^2$.

42. Find the remainder when $a^4 - 3\,a^3b + 2\,a^2b^2 - b^4$ is divided by $a^2 - ab + 2\,b^2$.

43. If $a = 0,\ b = 1,\ c = -2,\ d = 3,$ find the value of

$$(3\,abc - 2\,bcd)\sqrt[3]{a^3bc - c^3bd + 3}.$$

44. Find an expression which will divide both $4\,x^2 + 3\,x - 10$ and $4\,x^3 + 7\,x^2 - 3\,x - 15$ without remainder.

45. Simplify $\dfrac{a + \dfrac{ab}{a - b}}{a^2 - \dfrac{2\,a^2b^2}{a^2 + b^2}} \times \dfrac{\dfrac{1}{a^2} - \dfrac{1}{b^2}}{\dfrac{1}{a} - \dfrac{1}{b}}$.

46. Find the cube root of $8\,x^3 - 2\,x^2y + \dfrac{xy^2}{6} - \dfrac{y^3}{216}$.

47. Solve $\left.\begin{aligned} 9\,x + 8\,y &= 43\,xy \\ 8\,x + 9\,y &= 42\,xy \end{aligned}\right\}.$

48. Simplify $\dfrac{3}{x-4}-\dfrac{2}{x-5}-\dfrac{x-7}{(x-2)(x-3)}$.

49. Find the L. C. M. of
$8x^3+38x^2+59x+30$ and $6x^3-13x^2-13x+30$.

50. A boy spent half of his money in one shop, one-third of the remainder in a second, and one-fifth of what he had left in a third. He had 20 cents at last: how much had he at first?

51. Find the remainder when $x^7-10x^6+8x^5-7x^3+3x-11$ is divided by x^2-5x+4.

52. Simplify $4\left\{a-\frac{3}{2}\left(b-\frac{4c}{3}\right)\right\}\left\{\frac{1}{2}(2a-b)+2(b-c)\right\}$.

53. If $a=\frac{25}{16}$, $b=1$, $c=\frac{3}{4}$, prove that
$$(a-\sqrt{b})(\sqrt{a}+b)\sqrt{a-b}=\frac{3c^4}{\sqrt{a-c^2}}.$$

54. Find the L.C.M. of $x^2-7x+12$, $3x^2-6x-9$, and $2x^2-6x-8$.

55. Find the sum of the squares of $ax+by$, $bx-ay$, $ay+bx$, $by-ax$; and express the result in factors.

56. Solve $\dfrac{x}{6}+\dfrac{y}{4}=\dfrac{3x-5z}{4}=\dfrac{z}{8}+\dfrac{7y}{16}=1$.

57. Simplify $\dfrac{a^3+b^3}{a^4-b^4}-\dfrac{a+b}{a^2-b^2}-\dfrac{1}{2}\left\{\dfrac{a-b}{a^2+b^2}-\dfrac{1}{a-b}\right\}$.

58. Solve $x-\left(3x-\dfrac{2x+5}{10}\right)=\dfrac{1}{6}(2x+67)+\dfrac{5}{3}\left(1+\dfrac{x}{5}\right)$.

59. Add together the following fractions:
$$\frac{2}{x^2+xy+y^2},\ \frac{-4x}{x^3-y^3},\ \frac{x^2}{y^2(x-y)^2},\ \frac{-x^2}{x^3y-y^4}.$$

60. A man agreed to work for 30 days, on condition that for every day's work he should receive \$2.50, and that for every day's absence from work he should forfeit \$1.50; at the end of the time he received \$51: how many days did he work?

61. Divide $\dfrac{3x^5}{4}+27-\dfrac{43x^2}{4}-4x^4+\dfrac{77x^3}{8}-\dfrac{33x}{4}$ by $\dfrac{x^2}{2}+3-x$.

62. Find the value of
$$\frac{4y}{5}(y-x)-35\left[\frac{3x-4y}{5}-\frac{1}{10}\left\{3x-\frac{5}{7}(7x-4y)\right\}\right]$$
when $x=-\frac{1}{2}$ and $y=2$,

63. Simplify $\dfrac{10x-11}{3(x^2-1)}-\dfrac{10x-1}{3(x^2+x+1)}+\dfrac{x^2-2x+5}{(x^3-1)(x+1)}$.

64. Find the cube root of $\dfrac{a^3c^3}{b^3}x^6-\dfrac{3a^2c}{b}x^5+\dfrac{3ab}{c}x^4-\dfrac{b^3}{c^3}x^3$.

65. Solve $\dfrac{4x-17}{x-4}+\dfrac{10x-13}{2x-3}=\dfrac{8x-30}{2x-7}+\dfrac{5x-4}{x-1}$.

66. Find the factors of (i.) x^3+5x^2+x+5. (ii.) $x^2-2xy-323y^2$.

67. Solve $\left.\begin{aligned}\tfrac{1}{3}(x+y)+2z&=21\\ 3x-\tfrac{1}{2}(y+z)&=65\\ x+\tfrac{1}{2}(x+y-z)&=38\end{aligned}\right\}$.

68. Simplify $\dfrac{x+2y}{\frac{2}{7}x-y}-\dfrac{3x^2+63xy+70y^2}{2x^2+3xy-35y^2}$.

69. Find the square root of $-(3b-2c-2a)^3\{2(a+c)-3b\}$.

70. The united ages of a man and his wife are six times the united ages of their children. Two years ago their united ages were ten times the united ages of their children, and six years hence their united ages will be three times the united ages of the children: how many children have they?

71. Find the sum of

$x^2-3xy-\frac{2}{3}y^2$, $2y^2-\frac{2}{3}y^3+z^2$, $xy-\frac{1}{3}y^2+y^3$, and $2xy-\frac{1}{3}y^3$.

72. From $\{(a+b)(a-x)-(a-b)(b-x)\}$ subtract $(a+b)^2-2bx$.

73. If $a=5$, $b=4$, $c=3$, find the value of

$\sqrt[3]{6abc+(b+c)^3+(c+a)^3+(a+b)^3-(a+b+c)^3}$.

74. Find the factors of

(i.) $3x^3+6x^2-189x$. (ii.) $a^2+2ab+b^2+a+b$.

75. Solve $\left.\begin{aligned}px&=qy\\ (p+q)x-(q-p)y&=r\end{aligned}\right\}$.

76. Simplify $\dfrac{x+\frac{y}{2}}{2x^2+xy+\frac{y^2}{2}}-\dfrac{x^2-\frac{y^2}{2}}{4\left(x^3-\frac{y^3}{8}\right)}$.

77. Solve $\dfrac{x-7}{x+7}+\dfrac{1}{2(x+7)}=\dfrac{2x-15}{2x-6}$.

78. Reduce $\dfrac{x^4-x^2-2x+2}{2x^3-x-1}$ to its lowest terms.

79. Add together the fractions:

$$\frac{1}{2x^2-4x+2},\ \frac{1}{2x^2+4x+2},\ \text{and}\ \frac{1}{1-x^2}.$$

80. A number consists of three digits, the right-hand one being zero. If the left-hand and middle digits be interchanged, the number is diminished by 180; if the left-hand digit be halved, and the middle and right-hand digit be interchanged, the number is diminished by 336: find the number.

81. Divide $1-5x+\frac{152}{15}x^3-\frac{106}{225}x^4-\frac{28}{9}x^5$ by $1-x-\frac{14}{15}x^2$.

82. If $p=1$, $q=\frac{1}{2}$, find the value of

$$\frac{(p^2+q^2)-(p-q)\sqrt{p^2+2pq+q^2}}{2p+q-\{p-(q-p)\}}.$$

83. Multiply $\frac{3x^3}{2}-5x^2+\frac{x}{4}+9$ by $\frac{x^2}{2}-x+3$.

84. Find the L. C. M. of

$(a^2b-2ab^2)^2$, $2a^2-3ab-2b^2$, and $2(2a^2+ab)^2$.

85. Solve $\frac{2x+3}{x+1}=\frac{4x+5}{4x+4}+\frac{3x+3}{3x+1}$.

86. Reduce $\frac{5x^3-14x^2+16}{3x^3-2x^2+16x-48}$ to its lowest terms.

87. Find the square root of

$$4a^4+9\left(a^2+\frac{1}{a^2}\right)+12a(a^2+1)+18.$$

88. Solve

$$\left.\begin{aligned}\frac{x}{2a}+\frac{y}{3b}&=a+b\\ \frac{3x}{a}-\frac{2y}{b}&=6(b-a)\end{aligned}\right\}.$$

89. Multiply

$$3x+4y+\frac{11xy}{x-\frac{3}{2}y}\ \text{by}\ 10x-3y-\frac{11xy}{\frac{x}{4}+y}.$$

90. A bag contained ten dollars in dimes and quarters; after 17 dimes and 6 quarters were taken out, three times as many quarters as dimes were left: find the number of each coin.

91. Find the value of

$5(a-b)-2\{3a-(a+b)\}+7\{(a-2b)-(5a-2b)\}$,

when $a=-\frac{1}{9}b$.

92. Divide $3x^4-5x^3+7x^2-11x-13$ by $3x-2$.

93. Find the L. C. M. of

$15(p^3+q^3)$, $5(p^2-pq+q^2)$, $4(p^2+pq+q^2)$, and $6(p^2-q^2)$.

94. Resolve into factors:

(i.) $a^3-8\,b^{15}$. (ii.) $-x^2+2\,x-1+x^4$.

95. Solve $\dfrac{x+a}{x+b}=\dfrac{x+3\,a}{x+a+b}$.

96. Simplify

(i.) $\dfrac{35\,a^2b^2c^2-49\,b^3c^3}{65\,a^5bc-91\,a^3b^2c^2}$. (ii.) $\dfrac{y^4-7\,y^3+8\,y^2-12\,y}{2\,y^2-2\,y-60}$.

97. Solve
$$\left.\begin{aligned}7\,x-9\,y+4\,z&=16\\ \frac{x+y}{3}&=\frac{x+y+z}{2}\\ 2\,x-3\,y+4\,z-5&=0\end{aligned}\right\}.$$

98. Simplify $\dfrac{y^2-\dfrac{2\,y}{y-1}}{y^2-\dfrac{2\,y}{y+1}}\div\left(\dfrac{y^2-5\,y-6}{y^2-6\,y+5}\times\dfrac{y-2}{y+2}\right)$.

99. Find the square root of
$$\frac{4\,a^2-12\,ab-6\,bc+4\,ac+9\,b^2+c^2}{4\,a^2+9\,c^2-12\,ac}.$$

100. An express leaves New York at 3 P.M. and reaches Albany at 6; the ordinary train leaves Albany at 1.30 P.M. and arrives at New York at 6. If both trains travel uniformly, find the time when they will meet.

101. Solve (i.) $.\dot{6}\,x+.75\,x-.1\dot{6}=x-.58\dot{3}\,x+5$.

(ii.) $\dfrac{37}{x^2-5\,x+6}+\dfrac{4}{x-2}=\dfrac{7}{3-x}$.

102. Simplify (i.) $\dfrac{a+x}{a^2+x+ax^2}+\dfrac{a-x}{a^2-ax+x^2}+\dfrac{2\,x^3}{a^4+a^2x^2+x^4}$.

(ii.) $(1+x)^2\div\left\{1+\dfrac{x}{1-x+\dfrac{x}{1+x+x^2}}\right\}$.

103. Find the square root of
$$a^6+\frac{1}{a^6}-6\left(a^4+\frac{1}{a^4}\right)+15\left(a^2+\frac{1}{a^2}\right)-20\,;$$
also the cube root of the result.

104. Divide $1-2\,x$ by $1+3\,x$ to 4 terms.

105. I bought a horse and carriage for \$450; I sold the horse at a gain of 5 per cent, and the carriage at a gain of 20 per cent, making on the whole a gain of 10 per cent: find the original cost of the horse.

106. Find the divisor when $(4a^2 + 7ab + 5b^2)^2$ is the dividend, $8(a + 2b)^2$ the quotient, and $b^2(9a + 11b)^2$ the remainder.

107. Solve (i.) $5x(x-3) = 2(x-7)$.

$$\text{(ii.)}\quad \frac{1}{(x-1)(x-2)} + 6 = \frac{3}{x-2} + \frac{2}{x-1}.$$

108. If $x = a + b + \dfrac{(a-b)^2}{4(a+b)}$, and $y = \dfrac{a+b}{4} + \dfrac{ab}{a+b}$,
prove that $(x-a)^2 - (y-b)^2 = b^2$.

109. Find the square root of

$$49x^4 + \frac{1051x^2}{25} - \frac{14x^3}{5} - \frac{6x}{5} + 9.$$

110. Solve $\dfrac{a+x}{a^2 + ax + x^2} + \dfrac{a-x}{a^2 - ax + x^2} = \dfrac{3a}{x(a^4 + a^2x^2 + x^4)}$.

111. Subtract $\dfrac{x+3}{x^2 + x - 12}$ from $\dfrac{x+4}{x^2 - x - 12}$,
and divide the difference by $1 + \dfrac{2(x^2 - 12)}{x^2 + 7x + 12}$.

112. Find the H. C. F. and L. C. M. of

$2x^2 + (6a - 10b)x - 30ab$ and $3x^2 - (9a + 15b)x + 45ab$.

113. Solve (i.) $2cx^2 - abx + 2abd = 4cdx$.

$$\text{(ii.)}\quad \frac{x}{2(x+3)} - 2\tfrac{5}{24} = \frac{x^2}{x^2 - 9} - \frac{8x - 1}{4(x-3)}.$$

114. If $a = 1$, $b = 2$, $c = 3$, $d = 4$, find the value of

$$\frac{a^b + b^c + c^d}{b^a + c^b + d^c + (a+b)(b+c)} + 3(a^a + b^b + c^c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

115. I rode one-third of a journey at 10 miles an hour, one-third more at 9, and the rest at 8 miles an hour; if I had ridden half the journey at 10, and the other half at 8 miles per hour, I should have been half a minute longer on the way: what distance did I ride?

116. The product of two factors is $(3x + 2y)^3 - (2x + 3y)^3$, and one of the factors is $x - y$: find the other factor.

117. If $a + b = 1$, prove that $(a^2 - b^2)^2 = a^3 + b^3 - ab$.

118. Resolve into factors:

(i.) $x^3 + y^3 + 3xy(x+y)$. (ii.) $m^3 - n^3 - m(m^2 - n^2) + n(m-n)^2$.

119. Solve (i.) $\left.\begin{array}{r} x^3 - y^3 = 28 \\ x^2 + xy + y^2 = 7 \end{array}\right\}$. (ii.) $\left.\begin{array}{r} x^2 - 6xy + 11y^2 = 9 \\ x - 3y = 1 \end{array}\right\}$.

120. Find the square root of

$$(a - b)^4 - 2(a^2 + b^2)(a - b)^2 + 2(a^4 + b^4).$$

121. Simplify the fractions

(i.) $\dfrac{1}{a^2 - \dfrac{a^3 - 1}{a + \dfrac{1}{a + 1}}}$. (ii.) $\dfrac{\left(1 + \dfrac{1}{x}\right) \times \left(1 - \dfrac{1}{x}\right)^2}{x - \dfrac{1}{x}}$.

122. Find the H. C. F. of

$$a^2b + b^2c - abc - ab^2 \text{ and } ax^2 + ab - a^2 - bx^2.$$

123. A village had two-thirds of its voters Republicans: in an election 25 refused to vote, and 60 went over to the Democrats; the voters were now equal. How many voters were there altogether?

124. Solve (i.) $\dfrac{x^2}{a + b} + (a - b) = \dfrac{2ax}{a + b}$.

(ii.) $\dfrac{3}{x} + \dfrac{2}{y} = 6\left(\dfrac{1}{y} - \dfrac{1}{2x}\right) = 2$.

125. Simplify (i.) $\left(1 + \dfrac{y^2 + z^2 - x^2}{2yz}\right) \div \left(1 - \dfrac{x^2 + y^2 - z^2}{2xy}\right)$.

(ii.) $\dfrac{(x + 1)^3 - (x - 1)^3}{(x + 1)^4 - (x - 1)^4}$.

126. Divide

$$x^4 + (a - 1)x^3 - (2a + 1)x^2 + (a^2 + 4a - 5)x + 3a + 6$$

by $x^2 - 3x + a + 2$.

127. Resolve into factors:

(i.) $x^2 + 5xy - 24y^2 + x - 3y$. (ii.) $x^3 - \dfrac{4}{x}$.

128. Find the square root of $p^2 - 3q$ to three terms.

129. Solve (i.) $\dfrac{x - 5}{x - 6} - \dfrac{x - 6}{x - 7} = \dfrac{x - 1}{x - 2} - \dfrac{x - 2}{x - 3}$.

(ii.) $ax + 1 = by + 1 = ay + bx$.

130. Find the H. C. F. of

$$3x^2 + (4a - 2b)x - 2ab + a^2 \text{ and } x^3 + (2a - b)x^2 - (2ab - a^2)x - a^2b.$$

131. Simplify

(i.) $\dfrac{(x^a)^3}{x^{b+c}} \times \dfrac{(x^b)^3}{x^{c+a}} \times \dfrac{(x^c)^3}{x^{a+b}}$. (ii.) $x^{\frac{1}{2}}y^{\frac{1}{3}}\left(\dfrac{y^{\frac{1}{4}}}{x^{\frac{1}{6}}}\right)^2 \div \dfrac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}}$.

132. At a cricket match the contractor provided dinner for 27 persons, and fixed the price so as to gain $12\frac{1}{2}$ per cent upon his outlay. Six of the cricketers being absent, the remaining 21 paid the fixed price for their dinner, and the contractor lost \$3: what was the charge for the dinner?

133. Prove that $x(y+2)+\frac{x}{y}+\frac{y}{x}$ is equal to a, if

$$x=\frac{y}{y+1} \text{ and } y=\frac{a-2}{2}.$$

134. Find the cube root of

$$x^3-12x^2+54x-112+\frac{108}{x}-\frac{48}{x^2}+\frac{8}{x^3}.$$

135. Find the H. C. F. and L. C. M. of

$$x^3+2ax^2+a^2x+2a^3 \text{ and } x^3-2ax^2+a^2x-2a^3.$$

136. Simplify

(i.) $42\left\{\frac{4x-3y}{6}-\frac{3x-4y}{7}\right\}-56\left\{\frac{3x-2y}{7}-\frac{2x-3y}{8}\right\}.$

(ii.) $\frac{4b+a}{3b+a}+\frac{a-4b}{a-3b}+\frac{a^2-3b^2}{a^2-9b^2}.$

137. Resolve $4a^2(x^3+18ab^2)-(32a^5+9b^2x^3)$ into four factors.

138. Solve (i.) $5\sqrt{3x}-1=\sqrt{75x-29}.$

$$\text{(ii.) } \frac{xy}{x+y}=70, \quad \frac{xz}{x+z}=84, \quad \frac{yz}{y+z}=140.$$

139. Show that the difference between

$$\frac{x}{x-a}+\frac{x}{x-b}+\frac{x}{x-c} \text{ and } \frac{a}{x-a}+\frac{b}{x-b}+\frac{c}{x-c}$$

is the same whatever value x may have.

140. Multiply $x^{\frac{3}{2}}+2y^{\frac{3}{2}}+3z^{\frac{3}{2}}$ by $x^{\frac{3}{2}}-2y^{\frac{3}{2}}-3z^{\frac{3}{2}}$.

141. Walking $4\frac{1}{4}$ miles an hour, I start $1\frac{1}{2}$ hours after a friend whose pace is 3 miles an hour; how long shall I be in overtaking him?

142. Express in the simplest form

(i.) $(8^{\frac{2}{3}}+4^{\frac{3}{2}})\times 16^{-\frac{3}{4}}.$ (ii.) $\dfrac{\left\{9^n\cdot 3^2\times\frac{1}{3^{-n}}\right\}-27^n}{3^{3n}\times 9}.$

143. Find the square root of

$$\frac{x}{y}+\frac{y}{x}+3-2\sqrt{\frac{x}{y}}-2\sqrt{\frac{y}{x}}.$$

144. Simplify

(i.) $\left(\dfrac{x}{x-1}-\dfrac{1}{x+1}\right)\cdot\dfrac{x^3-1}{x^6+1}\cdot\dfrac{(x-1)^2(x+1)^2+x^2}{x^4+x^2+1}.$

(ii.) $\left\{\dfrac{a^4-y^4}{a^2-2ay+y^2}\div\dfrac{a^2+ay}{a-y}\right\}\times\left\{\dfrac{a^5-a^3y^2}{a^3+y^3}\div\dfrac{a^4-2a^3y+a^2y^2}{a^2-ay+y^2}\right\}.$

145. Find the value of

(i.) $\sqrt{8}+\sqrt{50}-\sqrt{18}+\sqrt{48}.$ (ii.) $\sqrt{35+14\sqrt{6}}.$

146. Solve (i.) $\dfrac{x-b}{x-a}-\dfrac{x-a}{x-b}=\dfrac{2(a-b)}{x-(a+b)}.$

(ii.) $\left.\begin{aligned}2x+3y&=1\tfrac{1}{2}\\4x^2+9xy+9y^2&=11\end{aligned}\right\}$

147. Show that

$$\frac{(a+b)^3-c^3}{(a+b)-c}+\frac{(b+c)^3-a^3}{b+c-a}+\frac{(c+a)^3-b^3}{c+a-b}$$

is equal to $2(a+b+c)^2+a^2+b^2+c^2.$

148. Divide $a-x+4a^{\frac{1}{4}}x^{\frac{3}{4}}-4a^{\frac{1}{2}}x^{\frac{1}{2}}$

by $a^{\frac{1}{2}}+2a^{\frac{1}{4}}x^{\frac{1}{4}}-x^{\frac{1}{2}}.$

149. Find the square root of

$$(a-1)^4+2(a^4+1)-2(a^2+1)(a-1)^2.$$

150. How much are pears a gross when 12 more for a dollar lowers the price five cents a dozen?

151. Show that if a number of two digits is six times the sum of its digits, the number formed by interchanging the digits is five times their sum.

152. Find the value of

$$\frac{1}{(a-b)(b-c)}-\frac{1}{(b-c)(a-c)}-\frac{1}{(c-a)(b-a)}.$$

153. Multiply

$$3+5x-\frac{12+41x+36x^2}{4+7x}\text{ by }5-2x+\frac{26x-8x^2-14}{3-4x}.$$

154. If $x-\dfrac{1}{x}=1$, prove that $x^2+\dfrac{1}{x^2}=3$, and $x^3-\dfrac{1}{x^3}=4.$

155. Solve (i.) $\dfrac{3x}{11}+\dfrac{23}{x+4}=\dfrac{1}{3}(x+5).$

(ii.) $\left.\begin{aligned}2x^2-3y^2&=23\\2xy-3y^2&=3\end{aligned}\right\}.$

156. Simplify (i.) $1\frac{3}{5}\sqrt{20} - 3\sqrt{5} - \sqrt{\frac{1}{5}}$. (ii.) $\frac{\sqrt{x}}{y^{-\frac{1}{3}}}\left(\frac{\sqrt[4]{y}}{x^{\frac{1}{6}}}\right) \div \frac{y^{-\frac{1}{4}}}{x^{\frac{1}{4}}}$.

157. Find the H. C. F. of $(p^2-1)x^2+(3p-1)x-p(p-1)$ and $p(p+1)x^2-(p^2-2p-1)x-(p-1)$.

158. Reduce to its simplest form

$$\frac{ax+\frac{a}{y}}{x-\frac{1}{y}} \times \frac{x^2+\frac{1}{y^2}}{bx^2-\frac{b}{y^2}} \times \frac{\frac{1}{5}(xy-1)^2}{\frac{1}{3}(x^4y^4-1)}.$$

159. Find the square root of

(i.) $1-2^{2n+1}+4^{2n}$. (ii.) $9^n-2\cdot 6^n+4^n$.

160. A clock gains 4 minutes a day. What time should it indicate at 6 o'clock in the morning in order that it may be right at 7.15 P.M. on the same day?

161. If $x=2+\sqrt{2}$, find the value of $x^2+\frac{4}{x^2}$.

162. Solve (i.) $\frac{\sqrt{x}+a}{\sqrt{x}-b}=\frac{\sqrt{x}-a}{\sqrt{x}}$. (ii.) $\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}=3$.

163. Simplify $\frac{a^2}{(b-a)(c-a)}+\frac{b^2}{(c-b)(a-b)}+\frac{c^2}{(a-c)(b-c)}$.

164. Find the product of $\frac{1}{5}\sqrt{5}$, $\frac{1}{2}\sqrt[3]{2}$, $\sqrt[6]{80}$, $\sqrt[3]{5}$, and divide

$$\frac{8-4\sqrt{5}}{\sqrt{5}+1} \text{ by } \frac{3\sqrt{5}-7}{5+\sqrt{7}}.$$

165. Resolve $9x^6y^2-576y^2-4x^8+256x^2$ into six factors.

166. Simplify (i.) $\frac{1-\frac{a^2}{(x+a)^2}}{(x+a)(x-a)} \div \frac{x(x+2a)}{(x^2-a^2)(x+a)^2}$.

(ii.) $\frac{6x^2y^2}{m+n} \div \left[\frac{3(m-n)x}{7(r+s)} \div \left\{\frac{4(r-s)}{21xy^2} \div \frac{r^2-s^2}{4(m^2-n^2)}\right\}\right]$.

167. Simplify (i.) $\left(a^{1+\frac{q}{p}}\right)^{\frac{p}{p+q}} \div \sqrt[p]{\frac{a^{2p}}{(a^{-1})^{-p}}}$.

(ii.) $\sqrt{14-\sqrt{132}}$.

168. Find the H. C. F. and L. C. M. of

$20x^4+x^2-1$, $25x^4+5x^3-x-1$, $25x^4-10x^2+1$.

169. Solve (i.) $a + x + \sqrt{2ax + x^2} = b.$

(ii.) $x + 9\frac{5}{8} + \dfrac{1}{\dfrac{x}{7} + \dfrac{11}{8}} = 8.$

170. The price of photographs is raised \$ 3 per dozen, and customers consequently receive ten less than before for \$ 5: what were the prices charged?

171. If $\left(a + \frac{1}{a}\right)^2 = 3$, prove that $a^3 + \frac{1}{a^3} = 0.$

172. Find the value of

$$\frac{x + 2a}{2b - x} + \frac{x - 2a}{2b + x} + \frac{4ab}{x^2 - 4b^2} \text{ when } x = \frac{ab}{a + b}.$$

173. Reduce to fractions in their lowest terms

(i.) $\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \div \left(\frac{x + y + z}{x^2 + y^2 + z^2 - xy - yz - zx} - \frac{1}{x + y + z}\right) + 1.$

(ii.) $\left(1 - \frac{56}{x + 4} + \frac{42}{x + 3}\right)\left(1 + \frac{56}{x - 4} - \frac{42}{x - 3}\right).$

174. Express as a whole number

$$(27)^{\frac{2}{3}} + (16)^{\frac{3}{4}} - \frac{2}{(8)^{-\frac{2}{3}}} + \frac{\sqrt[5]{2}}{(4)^{-\frac{2}{5}}}.$$

175. Simplify

(i.) $\frac{n}{1 - x^n} + \frac{n}{1 - x^{-n}}.$ (ii.) $\sqrt[4]{97 - 56\sqrt{3}}.$

176. Solve (i.) $\frac{x - 4a}{x - 3a} + \frac{x - 5a}{x - 4a} = \frac{x + 6a}{x - 4a} + \frac{x + 5a}{x - 3a}.$

(ii.) $\left.\begin{aligned} 3x^2 + xy + 3y^2 &= 8\tfrac{1}{4} \\ 8x^2 - 3xy + 8y^2 &= 17\tfrac{3}{4} \end{aligned}\right\}.$

177. Find the square root of $\frac{a^2x^2 + 2ab^2x^3 + b^4x^4}{a^{2m} + 2a^mx^n + x^{2n}}.$

178. Simplify

(i.) $\frac{b}{\sqrt{a}} \times \sqrt[3]{ac} \times \frac{\sqrt[4]{c^3}}{\sqrt{b}} \times \frac{\sqrt{b^{-1}}}{a^{-\frac{1}{6}}}.$ (ii.) $\left\{\frac{(9^{n+\frac{1}{4}}) \times \sqrt{3 \times 3^n}}{3\sqrt{3^{-n}}}\right\}^{\frac{1}{n}}.$

179. A boat's crew can row 8 miles an hour in still water: what is the speed of a river's current if it take them 2 hours and 40 minutes to row 8 miles up and 8 miles down?

180. If $a = x^2 - yz$, $b = y^2 - zx$, $c = z^2 - xy$, prove that

$$a^2 - bc = x(ax + by + cz).$$

181. Find a quantity such that when it is subtracted from each of the quantities a, b, c, the remainders are in continued proportion.

182. Simplify (i.) $\left(x + y - \dfrac{1}{x + y - \dfrac{xy}{x+y}}\right) \times \dfrac{x^3 - y^3}{x^2 - y^2}$.

(ii.) $\dfrac{2(7x - 4)}{6x^2 - 7x + 2} + \dfrac{x - 10}{6x^2 - x - 2} - \dfrac{2(4x - 1)}{4x^2 - 1}$.

183. Find the sixth root of

$$729 - 2916x^2 + 4860x^4 - 4320x^6 + 2160x^8 - 576x^{10} + 64x^{12}.$$

184. Simplify

(i.) $\dfrac{1}{x + \sqrt{x^2 - 1}} + \dfrac{1}{x - \sqrt{x^2 - 1}}$.

(ii.) $\sqrt[4]{16} + \sqrt[3]{81} - \sqrt[3]{-512} + \sqrt[3]{192} - 7\sqrt[6]{9}$.

185. Solve (i.) $\dfrac{5}{6 - \dfrac{5}{6 - \dfrac{5}{6 - x}}} = x$.

(ii.) $\left.\begin{array}{r} x^2y^2 + 192 = 28xy \\ x + y = 8 \end{array}\right\}$.

186. Simplify

$$\frac{b - c}{a^2 - (b - c)^2} + \frac{c - a}{b^2 - (c - a)^2} + \frac{a - b}{c^2 - (a - b)^2}.$$

187. Solve (i.) $x - 15\frac{3}{4} + \dfrac{5}{x - 15\frac{3}{4}} = 6$.

(ii.) $2(x + y^{-1}) = 3(x^{-1} - y) = 4$.

188. If $xy = ab(a + b)$ and $x^2 - xy + y^2 = a^3 + b^3$, prove that

$$\left(\frac{x}{a} - \frac{y}{b}\right)\left(\frac{x}{b} - \frac{y}{a}\right) = 0.$$

189. Find the H. C. F. of

$$(2a^2 - 3a - 2)x^2 + (a^2 + 7a + 2)x - a^2 - 2a$$

and

$$(4a^2 + 4a + 1)x^2 - (4a^2 + 2a)x + a^2.$$

190. Multiply $\sqrt{2x} + \sqrt{2(2x - 1)} - \dfrac{1}{\sqrt{2x}}$

by $\dfrac{1}{\sqrt{2x}} + \sqrt{2(2x - 1)} - \sqrt{2x}$.

191. Divide $a^4b^2 + b^4c^2 + c^4a^2 - a^2b^4 - b^2c^4 - c^2a^4$

by $a^2b + b^2c + c^2a - ab^2 - bc^2 - ca^2$.

192. Simplify

(i.) $\dfrac{7}{2(x+1)} - \dfrac{1}{6(x-1)} - \dfrac{10x-1}{3(x^2+x+1)}$.

(ii.) $\left\{\dfrac{\sqrt{x+a}}{\sqrt{x-a}} - \dfrac{\sqrt{x-a}}{\sqrt{x+a}}\right\} \times \dfrac{\sqrt{x^3-a^3}}{\sqrt{(x+a)^2-ax}}$.

193. If p be the difference between any quantity and its reciprocal, q the difference between the square of the same quantity and the square of its reciprocal, show that

$$p^2(p^2+4) = q^2.$$

194. A man started for a walk when the hands of his watch were coincident between three and four o'clock. When he finished, the hands were again coincident between five and six o'clock. What was the time when he started, and how long did he walk?

195. If n be an integer, show that $7^{2n+1} + 1$ is always divisible by 8.

196. Simplify $$\frac{\left(p+\frac{1}{q}\right)^p\left(p-\frac{1}{q}\right)^q}{\left(q+\frac{1}{p}\right)^p\left(q-\frac{1}{p}\right)^q}.$$

197. Find the value of

(i.) $\dfrac{7+3\sqrt{5}}{7-3\sqrt{5}} + \dfrac{7-3\sqrt{5}}{7+3\sqrt{5}}$.

(ii.) $\dfrac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}$ when $x = \dfrac{2b}{b^2+1}$.

198. If $a + b + c + d = 2s$, prove that

$$4(ab+cd)^2 - (a^2+b^2-c^2-d^2)^2 = 16(s-a)(s-b)(s-c)(s-d).$$

199. A man buys a number of articles for \$5, and sells for \$5.40 all but two at 5 cents apiece more than they cost; how many did he buy?

200. Find the square root of

$$2(81x^4 + y^4) - 2(9x^2+y^2)(3x-y)^2 + (3x-y)^4.$$

201. If $x : a :: y : b :: z : c$, prove that

$$(bc+ca+ab)^2(x^2+y^2+z^2) = (bz+cx+ay)^2(a^2+b^2+c^2).$$

202. If a man save \$10 more than he did the previous year, and if he saved \$20 the first year, in how many years will his savings amount to \$1700?

203. Given that 4 is a root of the quadratic $x^2 - 5x + q = 0$, find the value of q and the other root.

204. A person having 7 miles to walk increases his speed one mile an hour after the first mile, and finds that he is half an hour less on the road than he would have been had he not altered his rate. How long did he take?

205. If $(a+b+c)x = (-a+b+c)y = (a-b+c)z = (a+b-c)w$,

show that $$\frac{1}{y} + \frac{1}{z} + \frac{1}{w} = \frac{1}{x}.$$

206. Find a Geometrical Progression of which the sum of the first two terms is $2\frac{2}{3}$, and the sum to infinity $4\frac{1}{6}$.

207. Simplify $$\frac{\left(1+\frac{x}{y}\right)^m\left(1-\frac{y}{x}\right)^n}{\left(1+\frac{y}{x}\right)^n\left(1-\frac{x}{y}\right)^m}.$$

208. A man has a stable containing 10 stalls; in how many ways could he stable 5 horses?

209. In boring a well 400 feet deep the cost is 27 cents for the first foot and an additional cent for each subsequent foot; what is the cost of boring the last foot, and also of boring the entire well?

210. If a, b are the roots of $x^2 + px + q = 0$, show that p, q are the roots of the equation
$$x^2 + (a + b - ab)x - ab(a + b) = 0.$$

211. Extract the square root of $7 - 30\sqrt{-2}$.

212. If $\dfrac{x+z}{y} = \dfrac{z}{x} = \dfrac{x}{z-y}$, determine the ratios $x : y : z$.

213. If a, b, c are in H. P., show that
$$\left(\frac{3}{a}+\frac{3}{b}-\frac{2}{c}\right)\left(\frac{3}{c}+\frac{3}{b}-\frac{2}{a}\right)+\frac{9}{b^2}=\frac{25}{ac}.$$

214. Find the number of permutations which can be made from all the letters of the words

(i.) *Consequences*, (ii.) *Acarnania.*

215. Expand by the Binomial Theorem $(2a - 3x)^5$; and find the numerically greatest term in the expansion of $(1 + x)^n$, if $x = \frac{3}{5}$, and $n = 7$.

216. When $x = \dfrac{\sqrt{3}}{4}$, find the value of
$$\frac{1+2x}{1+\sqrt{1+2x}} + \frac{1-2x}{1-\sqrt{1-2x}}.$$

217. Simplify $\dfrac{x^2-bc}{(a-b)(a-c)}+\dfrac{x^2-ca}{(b-c)(b-a)}+\dfrac{x^2-ab}{(c-a)(c-b)}.$

218. Solve the equations

(i.) $(x^2-5x+2)^2=x^2-5x+22.$

(ii.) $\left(x^2+\dfrac{1}{x^2}\right)^2+4\left(x^2+\dfrac{1}{x^2}\right)=12.$

219. Prove that

$$(y-z)^3+(x-y)^3+3(x-y)(x-z)(y-z)=(x-z)^3.$$

220. Out of 16 consonants and 5 vowels, how many words can be formed each containing 4 consonants and 2 vowels?

221. If $b-a$ is a harmonic mean between $c-a$ and $d-a$, show that $d-c$ is a harmonic mean between $a-c$ and $b-c$.

222. In how many ways may 2 red balls, 3 black, 1 white, 2 blue be selected from 4 red, 6 black, 2 white, and 5 blue; and in how many ways may they be arranged?

223. The sum of a certain number of terms of an arithmetical series is 36, and the first and last of these terms are 1 and 11 respectively: find the number of terms, and the common difference of the series.

224. Expand by the Binomial Theorem

(i.) $\left(2-\dfrac{3a}{4}\right)^5.$ (ii.) $(1-\frac{2}{3}x)^{\frac{3}{2}}$ to five terms.

225. Solve

$$x^2-xy+x=35.$$
$$xy-y^2+y=15.$$

226. Simplify $\dfrac{2\sqrt{10}}{3\sqrt{27}}\times\dfrac{15\sqrt{21}}{4\sqrt{15}}\div\dfrac{5\sqrt{14}}{7\sqrt{48}}$ and find the value of $\dfrac{1}{3\sqrt{5}-6}$, given that $\sqrt{5}=2.236$.

227. By the Binomial Theorem find the cube root of 128 to six places of decimals.

228. There are 9 books, of which 4 are Greek, 3 are Latin, and 2 are English; in how many ways could a selection be made so as to include at least one of each language?

229. Simplify

(i.) $\dfrac{\sqrt{45x^3}-\sqrt{80x^3}+\sqrt{5a^2x}}{a-x}.$

(ii.) $\left\{\dfrac{x^{\frac{1}{2}}+x^{-\frac{1}{2}}}{x^2-x+1}-\dfrac{x^{\frac{1}{2}}-x^{-\frac{1}{2}}}{x^2+x+1}\right\}\div\left\{\dfrac{x^{\frac{1}{2}}+2x^{-\frac{1}{2}}}{x^3-1}-\dfrac{x^{\frac{1}{2}}-2x^{-\frac{1}{2}}}{x^3+1}\right\}.$

230. (i.) Form the quadratic equation whose roots are $5 \pm \sqrt{6}$.

(ii.) If the roots of $x^2 - px + q = 0$ are two consecutive integers, prove that $p^2 - 4q - 1 = 0$.

231. Solve $x^3 + 1 = 81(y^2 + y)$; $x^2 + x = 9(y^3 + 1)$.

232. Find $\log_{16} 128$, $\log_4 \sqrt{128}$, $\log_2 \frac{1}{4}$; and having given

$$\log 2 = .3010300 \text{ and } \log 3 = .4771213,$$

find the logarithm of .00001728.

233. A and B start from the same point, B five days after A; A travels 1 mile the first day, 2 miles the second, 3 miles the third, and so on; B travels 12 miles a day. When will they be together? Explain the double answer.

234. Solve the equations:

(i.) $2^x = 8^{y+1}$, $9^y = 3^{x-9}$.

(ii.) $z^x = y^{2x}$, $2^z = 2 \times 4^x$, $x + y + z = 16$.

235. The sum of the first 10 terms of an arithmetical series is to the sum of the first 5 terms as 13 is to 4; find the ratio of the first term to the common difference.

236. Find the greatest term in the expansion of $(1-x)^{-\frac{4}{3}}$ when $x = \frac{12}{13}$.

237. Five gentlemen and one lady wish to enter an omnibus in which there are only three vacant places; in how many ways can these places be occupied (1) when there is no restriction, (2) when one of the places is to be occupied by the lady?

238. (i.) Given $\log 2 = .301030$, $\log 3 = .477121$, and $\log 7 = .845098$, find the logarithms of .005, 6.3, and $(\frac{49}{216})^{\frac{1}{3}}$.

(ii.) Find x from the equation $18^{8-4x} = (54\sqrt{2})^{3x-2}$.

239. If P and Q vary respectively as $y^{\frac{1}{2}}$ and $y^{\frac{1}{3}}$ when z is constant, and as $z^{\frac{1}{2}}$ and $z^{\frac{1}{3}}$ when y is constant, and if $x = P + Q$, find the equation between x, y, z; it being known that when $y = z = 64$, $x = 12$; and that when $y = 4z = 16$, $x = 2$.

240. Simplify

$$\log \tfrac{133}{65} + 2 \log \tfrac{13}{7} - \log \tfrac{143}{90} + \log \tfrac{77}{171}.$$

241. If the number of permutations of n things 4 at a time is to the number of combinations of $2n$ things 3 at a time as 22 to 3, find n.

242. If $\dfrac{1}{a} + \dfrac{1}{c} = \dfrac{1}{2b - a} + \dfrac{1}{2b - c}$, prove that $2b$ is either the arithmetic mean between $2a$ and $2c$, or the harmonic mean between a and c.

243. If ${}^nC_r$ denote the number of combinations of n things taken r together, prove that

$$ {}^{n+2}C_{r+1} = {}^nC_{r+1} + {}^nC_{r-1} + (2 \times {}^nC_r). $$

244. Find (i.) the characteristic of log 54 to base 3.

(ii.) $\log_{10}(.0125)^{\frac{1}{3}}$. (iii.) the number of digits in 3^{45}.

Given $\log_{10} 2 = .30103$, $\log_{10} 3 = .47712$.

245. Write the $(r+1)$th term of $(2\,ax^2 - x^3)^{\frac{5}{3}}$, and express it in its simplest form.

246. At a meeting of a debating society there were 9 speakers; 5 spoke for the affirmative, and 4 for the negative. In how many ways could the speeches have been made, if a member of the affirmative always spoke first, and the speeches were alternately for the affirmative and the negative?

247. Form the quadratic equation whose roots are

$$ a + b + \sqrt{a^2 + b^2} \text{ and } \frac{2\,ab}{a + b + \sqrt{a^2 + b^2}}. $$

248. A point moves with a speed which is different in different miles, but invariable in the same mile, and its speed in any mile varies inversely as the number of miles travelled before it commences this mile. If the second mile be described in 2 hours, find the time taken to describe the nth mile.

249. Solve the equations:

(i.) $x^2(b-c) + ax(c-a) + a^2(a-b) = 0.$

(ii.) $(x^2 - px + p^2)(qx + pq + p^2) = qx^3 + p^2q^2 + p^4.$

250. Prove by the Binomial Theorem that $\sqrt{8}$ is the value, to infinity, of

$$ 1 + \frac{3}{4} + \frac{3 \cdot 5}{4 \cdot 8} + \frac{3 \cdot 5 \cdot 7}{4 \cdot 8 \cdot 12} + \cdots. $$

251. A and B run a mile race. In the first heat A gives B a start of 11 yards and beats him by 57 seconds; in the second heat A gives B a start of 81 seconds and is beaten by 88 yards: in what time could each run a mile?

252. A train, an hour after starting, meets with an accident which detains it an hour, after which it proceeds at three-fifths of its former rate and arrives 3 hours after time; but had the accident happened 50 miles farther on the line, it would have arrived $1\frac{1}{2}$ hours sooner: find the length of the journey.

253. Expand for 4 terms by the method of Undetermined Coefficients $\dfrac{2}{3x^2 - 2x^3}$.

254. A body of men were formed into a hollow square, three deep, when it was observed that with the addition of 25 to their number a solid square might be formed, of which the number of men in each side would be greater by 22 than the square root of the number of men in each side of the hollow square: required the number of men.

255. Expand into a series $\sqrt{a^2 + b^2}$.

256. Solve the equation $\sqrt{2x-1} + \sqrt{3x-2} = \sqrt{4x-3} + \sqrt{5x-4}$.

257. Separate $\dfrac{7x^2 + 22x + 5}{(x+3)(x^2-1)}$ into partial fractions.

258. Solve the equation $x^4 - 5x^2 - 6x - 5 = 0$.

259. Find the generating function of

$$1 + 5x + 7x^2 + 17x^3 + 31x^4 + \cdots.$$

260. Separate $\dfrac{3x - x^2 - 4}{(x^2+1)(x^2-x-2)}$ into partial fractions.

261. Solve the equation $x^4 + 3x^2 = 16x + 60$.

262. Express $\frac{763}{396}$ as a continued fraction and find the fourth convergent.

263. What is the sum of n terms of the series 1, 8, 27, 64, $\cdots$?

264. The sum of 6 terms of the series $1 - x\sqrt{-1} - x^2 + \cdots$ is equal to 65 times the sum to infinity; find x.

265. Convert $2\sqrt{5}$ into a continued fraction.

266. Find limits of the error when $\frac{916}{191}$ is taken for $\sqrt{23}$.

267. Sum to infinity the series

$$3 - x - 2x^2 - 16x^3 - 28x^4 - 676x^5 + \cdots.$$

268. Find value of $\dfrac{1}{3+}\,\dfrac{1}{2+}\,\dfrac{1}{1+}\,\dfrac{1}{3+}\,\dfrac{1}{2+}\,\dfrac{1}{1+}\cdots$.

269. Solve, by Cardan's Method, the equation

$$x^3 - 30x + 133 = 0.$$

270. Solve the equation $x^3 - 13x^2 + 15x + 189 = 0$, having given that one root exceeds another root by 2.

271. Solve the equation $x^4 - 4x^2 + 8x + 35 = 0$, having given that one root is $2 + \sqrt{-3}$.

272. Sum to infinity the series

$$4 - 9x + 16x^2 - 25x^3 + 36x^4 - 49x^5 + \cdots.$$

273. Solve the equation $x^4 - 12x^3 + 47x^2 - 72x + 36 = 0.$

274. Solve the equation $\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0.$

275. Solve the equation $2x^5 + x^4 + x + 2 = 12x^3 + 12x^2.$

276. Given $\log 2 = .30103$, and $\log 3 = .47712$, solve the equations:

(i.) $6^x = \frac{10}{3} - 6^{-x}.$ (ii.) $\sqrt{5^x} + \sqrt{5^{-x}} = \frac{29}{10}.$

277. Find the value of $1 + \frac{1}{3+}\,\frac{1}{2+}\,\frac{1}{3+}\,\frac{1}{2+}\cdots$ in the form of a quadratic surd.

278. Separate $\dfrac{x^3 + 7x^2 - x - 8}{(x^2 + x + 1)(x^2 - 3x - 1)}$ into partial fractions.

279. Find the general term when $\dfrac{3x-8}{x^2-4x-4}$ is expanded in ascending powers of x.

280. Solve the equations:

(i.) $\dfrac{\sqrt[3]{x+y} - \sqrt[3]{x-y}}{\sqrt[3]{x+y} + \sqrt[3]{x-y}} = 3,\ x^2 + y^2 = 65.$

(ii.) $\sqrt{2x^2 + 1} + \sqrt{2x^2 - 1} = \dfrac{2}{\sqrt{3 - 2x^2}}.$

(iii.) $x^5 - 4x^4 - 10x^3 + 40x^2 + 9x - 36 = 0.$

Date Due

DEC 13 '49	OCT 17 '72		
JAN 31 '51	DEC 15 '72		
JUN 11 '5[illegible]	OCT 2[illegible] 1976		
MAY 25 '5[illegible]	NOV 2[illegible] 1976		
MAR 17 '57	[illegible] 1977		
MAY 26 '58	[illegible]		
[illegible] 11 '59	1984 9 2 [illegible]		
NOV 21 '62	NOV. 27, 1984 B.P.		
DEC 1[illegible]	(ca[illegible]thocated)		
NOV 30 '6[illegible]	Dec. 9 – SEP 25 1990		
[illegible]			
FEB [illegible] '66			
[illegible] 2[illegible] '66			
OCT 30 '67			
JAN [illegible] '69			
[illegible] '69			
OCT 24 '70			
APR 8 '71			

NOV 3 '71

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